

Wavelet domain linear inversion with application to well logging

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Summary

Solving linear inversion problems in geophysics is a major challenge when dealing with non-stationary data. Certain non-stationary data sets can be shown to lie in Sobolev function spaces and are characterized by their smoothness (differentiability). This information can be input into an inverse problem by posing the problem in the wavelet domain. Contrary to Fourier transforms, wavelets form an unconditional basis for Sobolev spaces, allowing for a new generation of linear inversion schemes which incorporate smoothness information more precisely. As an example inversion is performed on smoothed and subsampled well log data.

Introduction

Multiple length scale variability in earth properties is a predominant characteristic of sedimentary basins. Sedimentary records (well-log data) display variability on length scales ranging from sub-centimeter to many kilometers. Traditional approaches in linear inversion (such as least squares deconvolution or kriging) are often based on certain homogeneity (stochastic stationarity) assumptions, and may encounter difficulties when evident time/length scale complexity has long range correlations at many scales.

Multifractal analysis shows (Herrmann, 1998) that traditional assumptions about geophysical data, such as that it lies in an L_2 function space, are incorrect. This criterion has been historically used due to its mathematical ease of derivation and manipulation. We will discuss how to move beyond the traditional assumptions of data smoothness and stationarity in order to better characterize the visible complexity and incorporate this information into a linear inversion scheme for estimating unknown geophysical parameters. This involves posing the inverse problem in the wavelet domain.

The set up is as follows. First we will briefly review the discrete wavelet transform and its relation to Sobolev spaces. We then describe the formulation of a linear inverse problem in the wavelet domain in which we generalize the traditional L_2 -norm assumptions. Finally we apply the method to the simple inverse problem of inverting the effects of a well logging tool.

The Discrete Wavelet Transform

To implement our methods on a computer we need to deal with finite dimensional data sets. Thus it is necessary

to review the discrete wavelet transform (DWT). We follow the exposition in (Choi and Baraniuk, 1999). The DWT represents a 1-D continuous function $f(x)$ as scalar products of the function with shifted versions of a lowpass scaling function $\phi(x)$ and shifted and dilated versions of a bandpass wavelet function $\psi(x)$. For appropriate choices of ϕ and ψ the functions $\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k)$, and $\psi(x) = 2^{j/2}\psi(2^j x - k)$, $j, k \in \mathcal{Z}$, form an unconditional orthonormal basis in a Sobolev space (see below) and we have:

$$f(x) = \sum_k u_{j_0,k} \phi_{j_0,k}(x) + \sum_{j=j_0}^{\infty} \sum_k w_{j,k} \psi_{j,k}(x) \quad (1)$$

where

$$u_{j_0,k} = \int f(x) \phi_{j_0,k}^*(x) dx, \quad w_{j,k} = \int f(x) \psi_{j,k}^*(x) dx. \quad (2)$$

In reality the input to the DWT will be a discrete finite length column vector \mathbf{f} instead of a function $f(x)$. The DWT can still be performed but the analyzing scaling and wavelet functions become finite length vectors. We assume that the vector \mathbf{f} to be analyzed has been pre-filtered (Strang and Nguyen, 1997). We can then write the DWT as a matrix vector product

$$\mathbf{f} = \Phi \mathbf{u} + \Psi \mathbf{w}. \quad (3)$$

Where the columns of Φ are the shifted discrete scaling functions and the columns of Ψ are the shifted discrete wavelet functions. We can combine Φ and Ψ into one matrix \mathbf{W}^T and \mathbf{u} and \mathbf{w} into one vector $\tilde{\mathbf{f}}$. Due to the finite length of the vector \mathbf{f} we must handle edge effects in the DWT just as in the Fourier transform. The method we use is periodization (Strang and Nguyen, 1997). We refer to \mathbf{W}^T as the *inverse* DWT. Due to the orthogonality of our basis functions (the columns of \mathbf{W}^T) we have that \mathbf{W}^T is an orthogonal matrix and its inverse is its transpose, \mathbf{W} . This is the *forward* DWT. To summarize, the forward and inverse DWT of a vector \mathbf{f} are represented as $\tilde{\mathbf{f}} = \mathbf{W}\mathbf{f}$ and $\mathbf{f} = \mathbf{W}^T \tilde{\mathbf{f}}$, respectively. For the remainder of this abstract we deal exclusively with finite dimensional, discrete, random vectors.

Fractional Splines

Choosing which ϕ and ψ to use is an area of active research. In this work we deal exclusively with *fractional spline* wavelets and scaling functions (Unser and Blu,

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2000). Fractional splines can be constructed such that the wavelet transform acts as a *Karhunen-Loeve* transform for functions with an arbitrary smoothness, i.e. it diagonalizes the covariance matrix of a vector belonging to an arbitrary Sobolev space (see below):

$$\mathbf{D} = \mathbf{W}^T \mathbf{C} \mathbf{W}, \quad (4)$$

where \mathbf{D} is a diagonal matrix containing the variances of the wavelet coefficients and \mathbf{C} is covariance matrix of the state domain vector. For details on constructing a fractional spline wavelet transform see (Unser and Blu, 2000). For another discussion on the use of fractional spline wavelets see Herrmann in these proceedings.

Previous inversion work in the wavelet domain (Li et al., 1996) using other wavelet bases has led to approximate methods because in those cases the matrix \mathbf{W} does not diagonalize the covariance matrix.

Sobolev Spaces

A function space is a set (collection) of functions. Such a set is constructed by defining a norm (magnitude) over all possible functions. If this norm is finite for a particular function we say that the function belongs to that function space. The most common norms used are the L_p norms:

$$\|f(x)\|_{L_p} = \left[\int |f(x)|^p \right]^{1/p}$$

In this work we characterize a function as belonging to a particular *Sobolev* space, H_p^α . We choose this space because it contains many realistic functions, such as well data, and it is parameterized by only two numbers, p and α . p describes the distribution of energy in the wavelet coefficients while α roughly tells the number of derivatives that also have finite norm. The Sobolev norm is defined in the wavelet domain as:

$$\|f(x)\|_{H_p^\alpha} = \left[\sum_k |u_{j_0, k}|^p \right]^{1/p} + \left[\sum_{j \geq j_0} \sum_k 2^{j(\alpha + p/2 - 1)} |w_{j, k}|^p \right]^{1/p}. \quad (5)$$

$w_{j, k}$ and $u_{j_0, k}$ are the components of the vectors \mathbf{u} and \mathbf{w} defined above.

We can also write equation 5 in matrix notation:

$$\|\mathbf{f}\|_{H_p^\alpha} = [\mathbf{1}^T \mathbf{u}]^{1/p} + [\mathbf{1}^T \mathbf{\Omega} \mathbf{w}]^{1/p}.$$

The notation $|\cdot|^p$ corresponds to exponentiating each component of a vector to the p^{th} power. The $\mathbf{\Omega}$ matrix consists of the $2^{j(\alpha + \frac{1}{2} - \frac{1}{p})}$ terms along the diagonal and zeros everywhere else. The $\mathbf{1}^T$ vector performs the action of summing all components of the vector it operates on.

Linear inversion

We let \mathbf{m} represent the model parameters we wish to estimate. Let \mathbf{d} represent observed values of this data. We have a linear forward modelling operator in the form of a matrix, \mathbf{P} , that operates on \mathbf{m} . There is also possibly noise, \mathbf{n} , in the experiment. We have the following relationship:

$$\mathbf{d} = \mathbf{P} \mathbf{m} + \mathbf{n}. \quad (6)$$

The solution of an inverse problem involves minimizing

$$\|\mathbf{n}\|_{H_p^\alpha} = \|(\mathbf{d} - \mathbf{P} \mathbf{m})\|_{H_p^\alpha} \quad (7)$$

over all possible values of \mathbf{m} .

In many inverse problems, there is insufficient information in the data to allow a unique solution, i.e., there are an infinite number of solution vectors \mathbf{m} that satisfy equation 7. In order for there to be a unique solution we must provide some other constraint to the problem. This usually comes in the form of a smoothness constraint in which we minimize the norm of the model vector

$$\|\mathbf{m}\|_{H_p^\beta}. \quad (8)$$

In order to obtain a solution to an inverse problem we must find the model vector, $\hat{\mathbf{m}}$ that simultaneously minimizes a norm on the noise vector and the unknown model:

$$\hat{\mathbf{m}} = \min \left[\|(\mathbf{d} - \mathbf{P} \mathbf{m})\|_{H_p^\alpha} + \|\mathbf{m}\|_{H_p^\beta} \right]. \quad (9)$$

In a stochastic framework the second term in this equation is referred to as the prior.

To transform the inverse problem to the wavelet domain we change equation 6 to

$$\mathbf{d} = \mathbf{P} \mathbf{W}^T \mathbf{W} \mathbf{m} + \mathbf{n}, \quad (10)$$

where $\mathbf{W}^T \mathbf{W} = \mathbf{I}$, or,

$$\mathbf{d} = \tilde{\mathbf{P}} \tilde{\mathbf{m}} + \mathbf{n} \quad (11)$$

where $\tilde{\mathbf{P}} = \mathbf{P} \mathbf{W}^T$ is the wavelet transform of the each row of the matrix \mathbf{P} and $\tilde{\mathbf{m}} = \mathbf{W} \mathbf{m}$. This expresses the forward problem in terms of the wavelet coefficients.

Since we assume that the measurement noise is white and gaussian we set $\alpha = -0.5$ in equation 9. We will also set $p = 2$ in all computations for convenience. Thus the measurement noise lies in $H_2^{-0.5}$.

All the random vectors here are assumed to be gaussian. The prior probability distribution on the wavelet coefficients is then:

$$p(\tilde{\mathbf{m}}) = \frac{1}{(2\pi)^{\frac{N}{2}} |\mathbf{D}|^{\frac{1}{2}}} \exp\left[-\frac{1}{2} (\tilde{\mathbf{m}})^T \mathbf{D}^{-1} (\tilde{\mathbf{m}})\right] \quad (12)$$

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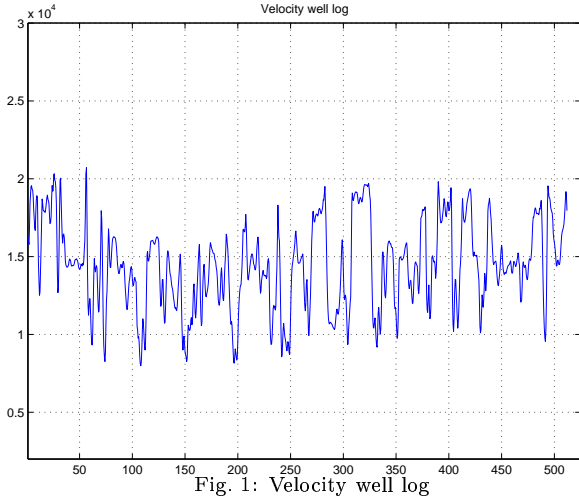


Fig. 1: Velocity well log

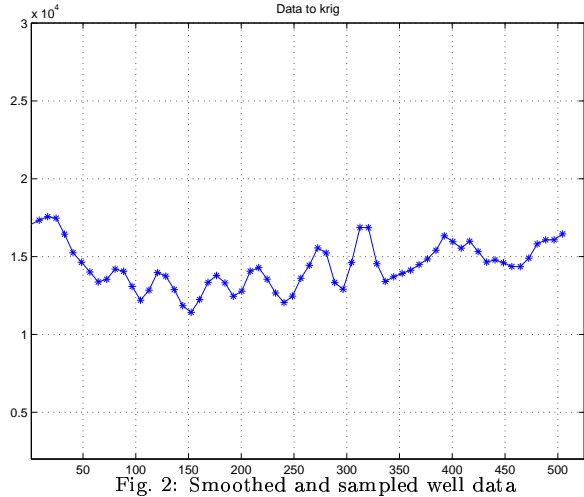


Fig. 2: Smoothed and sampled well data

where $\mathbf{D}^{-1} = \mathbf{\Omega}^T \mathbf{\Omega}$. The distribution of the measurement noise is

$$p(\mathbf{d}|\tilde{\mathbf{m}}) = \frac{1}{(2\pi)^{\frac{N}{2}}} \exp\left[-\frac{1}{2}(\mathbf{d} - \tilde{\mathbf{P}}\tilde{\mathbf{m}})^T \mathbf{C}_n^{-1}(\mathbf{d} - \tilde{\mathbf{P}}\tilde{\mathbf{m}})\right]. \quad (13)$$

where $\mathbf{C}_n^{-1} = \sigma^2 \mathbf{I}$ is the covariance function of the measurement noise.

Equation 13 is often called the “likelihood” function. Standard Bayesian inverse theory then defines the *posterior* probability distribution as

$$p(\tilde{\mathbf{m}}|\mathbf{d}) = k p(\mathbf{d}|\tilde{\mathbf{m}})p(\tilde{\mathbf{m}}) \quad (14)$$

Maximizing the probability in this expression corresponds to minimizing its exponent which is the Sobolev norm in equation 9.

Application to well logging

We shall now show an application in which we attempt to undo the bandlimiting effect that a logging tool has on a velocity field it is sampling. In this case the operator \mathbf{P} above is a cascade of two other operators \mathbf{P}_1 and \mathbf{P}_2 . \mathbf{P}_1 is a subset of the rows of the identity matrix that picks rows out the vector it operates on. \mathbf{P}_2 is a convolution matrix - a constant diagonal Toeplitz matrix which low-pass filters the vector it operates on. The matrix $\mathbf{P} = \mathbf{P}_1 \mathbf{P}_2$ is then a rectangular matrix comprised of a subset of the rows of the convolution matrix. The cascade of these two operator mimics the effect of a logging tool that continuously measures velocity in a bore hole with an array of sensors and then samples at regular time intervals.

In figure 1 we first show a typical velocity well log sampled at $0.5ft.$ intervals in the earth over roughly $500ft.$ We will assume this to be the “true” velocity vector and then then apply our logging tool to smooth and sample it. Our measured response of the velocity is shown in

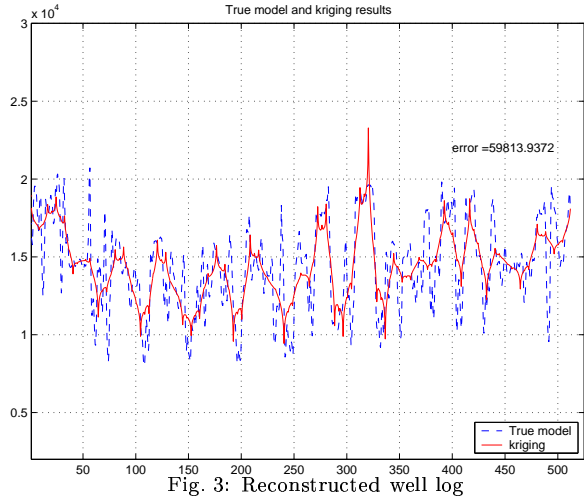


Fig. 3: Reconstructed well log

figure 2. The variance of the noise vector in this case is $\sigma^2 = 100ft.^2/s^2$. This is quite small and means that we assume our measuring tool to be accurate and noise free.

In order for inversion to proceed we need prior knowledge of the β parameter in equation 9 that determines the velocity field's smoothness. From previous analysis of similar wells we know that the exponent lies around $\beta = 0.1$. Using this in the inversion we obtain the results in figure 3. As can be seen, a great deal of the original detail can be reconstructed.

In essence, what has been done here is a joint deconvolution and kriging. Deconvolution in that the bandlimiting effects of a measurement have been undone. Kriging, because the data was subsampled and we were required to interpolate at unknown locations. These results foreshadow the application of this method to least-squares deconvolution of seismic traces and kriging of well logs at interwell locations.

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Conclusions

Formulating a linear inverse problem in the wavelet domain provides a flexible and easy way to incorporate smoothness information into an inverse problem. By using a correct fractional spline wavelet the wavelet transform whitens a random vector in a particular Sobolev space H_p^α . The results of the inversion on a simple problem are promising and future work will concentrate on least-squares seismic deconvolution and kriging.

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