

# THIN-FILM FLOWS WITH WALL SLIP: AN ASYMPTOTIC ANALYSIS OF HIGHER ORDER GLACIER FLOW MODELS

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## Summary

Free-surface thin film flows can principally be described by two types of models. Lubrication models assume that shear stresses are dominant in the force balance of the flow, and are appropriate where there is little or no slip at the base of the flow. Conversely, membrane or ‘free film’ models are appropriate in situations where there is rapid slip and normal (or extensional) stresses play a significant role in force balance. In some physical applications, notably in glaciology, both rapid and slow slip can occur within the same fluid film. In order to capture the dynamics of rapid and slow slip in a single model that describes the entire fluid film, a hybrid of membrane and lubrication models is therefore required. Several of these hybrid models have been constructed on an ad hoc basis in glaciology, where they are usually termed ‘higher order models’. Here we present a self-consistent asymptotic analysis of the most common of these models, due originally to Blatter. We show that Blatter’s model reproduces the solution to the underlying Stokes equations to second order in the film’s aspect ratio, regardless of the amount of slip at the base of the fluid. In doing so, we also construct asymptotic expansions for the Stokes equations to this order for shear-thinning power law fluids, paying particular attention to a high-viscosity boundary layer that develops at the free surface when there is little or no slip at the base. Lastly, we demonstrate that a depth-integrated hybrid model of comparable accuracy to Blatter’s model — which cannot be depth-integrated — can also be constructed, which we suggest as a viable tool for numerical simulations of thin films that contain both, slowly and rapidly sliding parts.

## 1. Introduction

Thin film flows are ubiquitous in engineering, geophysics, biology and elsewhere, and low aspect ratios are often the basis for simplified fluid dynamical models. One geophysical example is the flow of glaciers and ice sheets: over time scales longer than a few hours, ice creeps as a shear-thinning viscous fluid, and in almost all cases of interest, naturally occurring ice masses have low aspect ratios. For instance, the ice sheet covering Antarctica is several kilometres thick, but has a horizontal extent of several thousand kilometres, yielding an aspect ratio  $\varepsilon \approx 10^{-3}$ .

Motivated by the dynamics of glaciers and ice sheets, our aim here is to study the effect of wall slip on free surface thin film flows at low Reynolds number. Slip at the contact

between ice and glacier bed is a common phenomenon in glaciology, and complicates the task of modelling glaciers and ice sheets considerably. In the absence of wall slip, free surface thin film flows are typically well-described by a classical lubrication approximation (**1**, **2**), in which shearing across the film thickness balances a pressure gradient or component of body force in the plane of the film. The same approximation remains valid in the presence of a moderate amount of wall slip, which adds a component of velocity due to motion at the base of the film, but does not affect the stress field in the fluid at leading order. Large amounts of wall slip, on the other hand, can have a leading-order effect on the mechanics of the flow by introducing significant deviatoric normal stress (or ‘extensional stress’) gradients in the plane of the film. In that case, the classical lubrication approximation will fail, and a viscous membrane model becomes appropriate, which accounts for these normal stress gradients but ignores shearing across the film thickness. An extreme example of this behaviour is a so-called ‘free film’ (**3**), in which there is no friction at the base of the fluid film at all, and its flow is controlled entirely by extensional stresses.

Similar complications arise elsewhere in geophysical flows as well as in biological fluids: for instance, in the dynamics of cooling lava domes, a cold surface layer can be formed in which viscosity greatly exceeds that of the underlying lava. High velocities in the upper layer can then result from the rapid deformation of the much less viscous lava beneath, resulting in a scenario analogous to wall slip. If the viscosity contrast is sufficiently large, extensional stresses can then become important in the deformation of the upper layer in the way described above (**4**). Similarly, mucus transport in lungs often involves the deformation of a highly viscous mucus layer underlain by a much less viscous, watery pericillary liquid layer (**5**, **6**). Again, the much less viscous lower layer serves to create an effective wall slip, and extensional stresses can become important in the upper layer.

Different leading-order models therefore apply to thin films sliding slowly and rapidly. Both situations occur frequently in glacier and ice sheet dynamics, and often within the same ice mass. For instance, some glaciers *surge*, that is, they undergo a spatially-extended relaxation oscillation in which flow periodically switches from slow to fast sliding and back, with dramatic effects on glacier extent and thickness (**7**, **8**). Likewise, ice sheets often contain slowly-flowing parts that are well-described by a classical lubrication approximation (**9**, **10**, **11**) as well as more rapidly sliding ice streams and floating ice shelves that experience little or no traction at their surfaces, and are therefore described by a membrane-type model at leading order (**12**, **13**, **14**).

Crucially, the parts of an ice sheet or glacier that slide slowly or rapidly need not be fixed but can evolve over time (for instance due to temperature changes at the base of the ice facilitating or suppressing sliding), and similar concerns potentially apply to the lava and mucus flow examples mentioned above. An important issue in simulations of these flows is to develop an efficient way of dealing with these mechanically distinct regions. This is particularly relevant for instance in simulating ice sheet response to changing environmental forcings, in which the evolution of a large ice mass needs to be computed accurately over long timescales.

Free boundary methods have been suggested as an approach to tracking regions of slow and fast sliding within an ice sheet, so that the relevant thin film model can be applied in each region (**15**, **16**, **17**). This is, however, not a simple approach to implement numerically, and many ice sheet simulation codes instead use so-called higher order models (**18**, **19**). These are hybrid models that contain features of both, lubrication flow and viscous

membrane models, but which are simpler and less computationally expensive to solve than the full Stokes equations. Their purpose is to describe both, slow and fast sliding, using the same set of partial differential equations, so that a single piece of computer code can then describe the motion of the entire ice mass, rather than having to couple different types of models at moving boundaries within the ice flow domain.

To date, the construction of higher order ice flow models in the glaciological literature has largely proceeded on an *ad hoc* basis. Most higher order models are derived simply by retaining terms in the underlying Stokes equations that would ordinarily be omitted from a lubrication model, in the hope of capturing physics that is relevant to the flow being studied (**19**, **20**, **21**, **22**, **23**). Attempts have been made to assess the performance of these higher order models numerically relative to the underlying Stokes equations (**24**, **25**). What is still missing is a self-consistent theoretical basis for these models, along the lines of the asymptotic expansions that underlie the classical lubrication and viscous membrane approximations (**9**, **10**, **12**, **13**). We aim to provide such a theoretical basis in the present paper.

In the main part of this paper, we focus on the most widely used higher order model in glaciology, due to Blatter (**18**). Starting with the underlying Stokes equations, we show how Blatter's model can be derived by dropping higher order terms in the aspect ratio  $\varepsilon$  of the film. However, the aspect ratio of the film is not the only controlling parameter in the model, as a second parameter  $\lambda$  that describes the degree of slip at the base of the fluid also features. In order to demonstrate that Blatter's model remains valid regardless of the slip parameter regime, we show subsequently that the solution to Blatter's model reproduces the solution to the Stokes equations to  $O(\varepsilon^2)$ , regardless of the behaviour of the slip-related parameter  $\lambda$ . A sketch of this development may be found in (**26**), and we cast this work here in the context of standard perturbation methods.

The approach we take is to develop asymptotic expansions for the solution to Blatter's model up to  $O(\varepsilon^2)$ , and to show that these are also valid for the Stokes equations to the same order. This procedure is somewhat laborious because different regimes for the slip parameter  $\lambda$  have to be treated separately, but leads to a number of useful additional results.

Firstly, the asymptotic expansions to  $O(\varepsilon^2)$  for free surface thin film flows with arbitrary amounts of slip are of interest in their own right, regardless of their relevance to Blatter's model and the glaciological application in mind. This is particularly true in the case of slow sliding, in which we expect a classical lubrication approximation to apply at leading order. In keeping with the glaciological application, we model the fluid film in question as a shear-thinning material with a power law rheology. With such a rheology, viscosity becomes infinite at vanishing deviatoric stress, and this occurs in the standard lubrication approximation for a free surface flow at the upper surface of the fluid film. More precisely, viscosity becomes very large near the upper surface of the film, and this engenders high normal stresses even for relatively small strain rates at the surface. In essence, we see that viscosity stratification in thin-film flows of shear thinning materials can lead to extensional stresses becoming locally important even in the absence of wall slip: this is the case when low stresses in the shear thinning rheology lead to very high viscosities (**27**, **28**), as is the case with a power-law in which viscosity actually becomes infinite at vanishing stress.

To capture these high normal stresses and to investigate their effect on the rest of the thin film flow, a boundary layer treatment near the upper surface is necessary. In section 3.5, we develop the necessary boundary layer theory in the course of our analysis of Blatter's model,

extending previous results in Johnson & McMeeking (28). In its scope, this boundary layer analysis is also not dissimilar from the treatment of the ‘unyielded’ region near the surface of thin film flows with a Bingham rheology in Balmforth & Craster (29). As in their study of a viscoplastic flow, we find that a classical lubrication approximation must break down at the upper surface of our thin film flow with a shear-thinning power law rheology, but that it nevertheless does capture the correct dynamics for the film at leading order.

A second outcome of the lengthy asymptotic analysis in section 3 is that it lays the groundwork for considering other higher order models. By contrast with more classical thin-film approximations such as the lubrication approximation or viscous membrane models, Blatter’s model cannot be depth-integrated. From a computational perspective, this makes the model unappealing, as the entire fluid film domain needs to be resolved. For the purpose of numerical simulations, one would like to have a model that retains the advantages of Blatter’s model (namely that it describes both, fast and slow sliding) but which can also be depth-integrated. We turn to the construction of such a depth-integrated alternative in section 4, again building on some earlier arguments by Hindmarsh (26, 24).

Before we launch into the analysis below, it is worth pointing out two things. Firstly, we base our scalings and the subsequent analysis on the dynamics of a thin film spreading under its own weight, as is appropriate for an ice sheet. A simple adaptation is also possible to the case of a thin film flowing down an incline, driven by a downslope component of gravity, but we do not treat this case here. Secondly, we underline that our aim is not to develop higher order expansions to arbitrary order. We are only concerned with analysing thin film flows to  $O(\varepsilon^2)$ , and moreover we do not develop the asymptotic expansions below as a practical way of approximating thin film flow solutions. Instead, we are concerned primarily with justifying the use of Blatter’s model and other higher order models based on an asymptotic analysis. This, as well as our focus on wall slip, sets our work apart from the thin-film flow analysis for ice sheets in (30).

## 2. The basic flow model

For simplicity, we consider only the two-dimensional flow of an incompressible highly viscous fluid with a shear-thinning power-law rheology (known in glaciology as *Glen’s law*, see Paterson (31), chapter 5). A Cartesian coordinate system with the  $x$ -axis oriented horizontally and the  $z$ -axis vertically upwards will be used, in which the upper surface of the fluid will be denoted by  $z = h(x, t)$ , and the base of the fluid by  $z = b(x)$ . The velocity field in the fluid will be written in the form  $\mathbf{u} = (u, v)$ .

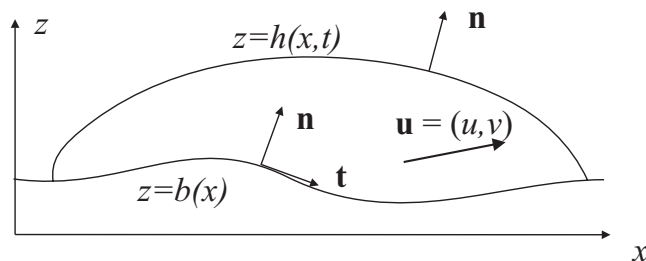


Fig. 1 Geometry of the thin-film flow problem.

In the fluid domain,  $b < z < h$ , force balance and conservation of mass require

$$\tau_{1,x} + \tau_{2,z} - p_x = 0, \quad (2.1)$$

$$-\tau_{1,z} + \tau_{2,x} - p_z - \rho g = 0, \quad (2.2)$$

$$u_x + v_z = 0, \quad (2.3)$$

where  $\rho$  is the density of the fluid and  $g$  acceleration due to gravity. The deviatoric stress tensor takes the form

$$\boldsymbol{\tau} = \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & -\tau_1 \end{pmatrix}, \quad (2.4)$$

and subscripts  $x$  and  $z$  denote partial derivatives, so  $u_x = \partial u / \partial x$  and  $\tau_{1,x} = \partial \tau_1 / \partial x$  etc. With a power law rheology, stress and strain rate are related through

$$u_x = A (\tau_1^2 + \tau_2^2)^{(n-1)/2} \tau_1, \quad (2.5)$$

$$u_z + v_x = 2A (\tau_1^2 + \tau_2^2)^{(n-1)/2} \tau_2, \quad (2.6)$$

where the parameters  $A$  and  $n$  are positive constants, used in accordance with the standard notation in glaciology (**31**). As stated before, we assume that the fluid is shear-thinning, so that  $n > 1$ . Note that, in keeping with standard notation in glaciology (and because it aids our later development) we have chosen to write strain rate as a function of stress in equations (2.5) and (2.6); more standard notation in fluid dynamics would simply invert these relations to express stress as a function of strain rate.<sup>†</sup> In terms of the glaciological application, we also assume that our ice mass is at a uniform temperature by using a constant value of  $A$ , as the viscosity of ice exhibits significant temperature dependence (**31**).

The base of the fluid is at a fixed elevation  $z = b(x)$ , and we assume zero normal velocity. To describe sliding over the substrate, we also assume a power-law friction law relating sliding velocity  $u_t = \mathbf{u} \cdot \mathbf{t}$  to basal shear stress  $\tau_{nt} = \mathbf{n} \cdot \boldsymbol{\tau} \cdot \mathbf{t}$  as  $\tau_{nt} = C |u_t|^{m-1} u_t$ . Here  $\mathbf{n} = (-b_x, 1) / \sqrt{1 + b_x^2}$  and  $\mathbf{t} = (1, b_x) / \sqrt{1 + b_x^2}$  are the upward-pointing normal and tangent to the bed, respectively, and  $C$  and  $m$  are positive constants. Explicitly, we have

$$v = ub_x, \quad (2.7)$$

$$\frac{-2b_x \tau_1 + (1 - b_x^2) \tau_2}{\sqrt{1 + b_x^2}} = C (1 + b_x^2)^{\frac{m}{2}} |u|^{m-1} u. \quad (2.8)$$

At the upper surface,  $z = h$ , there is no applied traction, which can be written in the form

$$(p - \tau_1) h_x + \tau_2 = 0, \quad (2.9)$$

$$-p - \tau_1 - \tau_2 h_x = 0. \quad (2.10)$$

In addition, the upper surface evolves according to a kinematic boundary condition with a source term  $a$ :

$$h_t + u h_x = v + a. \quad (2.11)$$

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<sup>†</sup> Specifically, if  $\tau_{ij}$  is shear stress and  $D_{ij}$  is strain rate, with  $D = \sqrt{D_{ij} D_{ij} / 2}$ , we could equally write  $\tau_{ij} = A^{-1/n} D^{-1+1/n} D_{ij}$ ; the above simply writes this in the form  $D_{ij} = A \tau^{n-1} \tau_{ij}$ , where  $\tau_{ij} = \sqrt{\tau_{ij} \tau_{ij} / 2}$ .

In glaciology,  $a$  represents snowfall ( $a > 0$ ) or melting ( $a < 0$ ), and is known as the *accumulation rate*. Integrating (2.3) from  $z = b$  to  $z = h$  and using (2.7) and (2.11) yields the depth-integrated mass conservation equation

$$h_t + q_x = a, \quad q = \int_b^h u dz. \quad (2.12)$$

### 2.1 Non-dimensionalization

The higher-order models we consider here are all thin-film models, but by contrast with other, more usual thin-film models such as the standard lubrication approximation (**9**, **10**, **11**) or viscous membrane models (**3**, **13**), they make no specific assumptions about the relative magnitude of sliding velocities compared with shearing across the film thickness. In order to develop these models, we first scale the basic Stokes flow model (2.1)–(2.11) in such a way as to account for arbitrary amounts of wall slip.

We assume that the geometry of the thin film is known, with a horizontal length scale  $[x]$  and a thickness scale  $[h]$ . This leaves scales for time  $[t]$ , velocities  $[u]$  and  $[v]$  and stresses  $[\tau_1]$  and  $[\tau_2]$  to be determined. We define these implicitly through the scale relations

$$[\tau_2] = \rho g [h]^2 / [x], \quad (2.13)$$

$$C [u]^m = [\tau_2] \quad (2.14)$$

$$[u] / [x] = A [\tau_1]^n, \quad (2.15)$$

$$[v] / [h] = [u] / [x], \quad (2.16)$$

$$[t] = [x] / [u]. \quad (2.17)$$

In choosing these relations, we have anticipated a number of balances. As is usual in models for thin films spreading under their own weight, we expect that the shear stress  $\tau_2$  scales with the driving stress  $-\rho g(h-b)h_x$  (for a film flowing down an inclined slope at angle  $\theta \gg [h]/[x]$ , there would be a downslope component of gravity  $\rho g \sin \theta$  in (2.1) and one would set  $[\tau_2] = \rho g \sin \theta$ , but the basic approach to the problem would remain the same as described in this paper). Importantly, we have also assumed that the velocity component parallel to the  $x$ -axis scales with sliding velocity through (2.14). This differs from the scalings used to derive for instance the standard lubrication approximation, which would assume that flow is mainly due to shearing across the fluid film and put  $[u] = A[\tau_2]^n[h]$  (see also (2.19) below). In addition, we anticipate that normal stress  $\tau_1$  is controlled by gradients in sliding velocity, as indicated by (2.15). The last two scale equations arise from incompressibility (2.3) and the depth-integrated mass conservation relation (2.12).

There are two independent parameters that determine the behaviour of the flow, the aspect ratio  $\varepsilon$  and a stress ratio  $\lambda$ :

$$\varepsilon = [h] / [x], \quad \lambda = [\tau_2] / [\tau_1]. \quad (2.18)$$

Importantly,  $\lambda$  can be thought of as a measure of slip. Let  $[u_d]$  be a scale for vertical shear, given by vertical shear stress  $[\tau_2]$  through the power-law rheology of the fluid:

$$[u_d] = A[\tau_2]^n[h]. \quad (2.19)$$

$\lambda$  can then be related to the slip ratio  $[u_d]/[u]$  through (32)

$$\frac{[u_d]}{[u]} = \lambda^n \varepsilon. \quad (2.20)$$

Hence large  $\lambda$  corresponds to significant vertical shearing (compared with sliding at the base), while small  $\lambda$  corresponds to a sliding-dominated flow. In other words, changes in  $\lambda$  correspond to different sliding regimes.

The model is scaled as follows:

$$\begin{aligned} h &= [h] h^*, & b &= [h] b^*, & z &= [h] z^*, & x &= [x] x^*, & t^* &= [t] t, & a &= [v] a^* \\ u &= [u] u^*, & v &= [v] v^*, & \tau_1 &= [\tau_1] \tau_1^*, & \tau_2 &= [\tau_2] \tau_2^*, & p &= \rho g [h] h^*. \end{aligned} \quad (2.21)$$

For simplicity, we immediately drop the asterisks on scaled variables. In scaled form, the model becomes the following. In  $b < z < h$ ,

$$\frac{\varepsilon}{\lambda} \tau_{1,x} + \tau_{2,z} - p_x = 0, \quad (2.22)$$

$$-\frac{\varepsilon}{\lambda} \tau_{1,z} + \varepsilon^2 \tau_{2,x} - p_z - 1 = 0, \quad (2.23)$$

$$u_x + v_z = 0, \quad (2.24)$$

$$u_x = (\tau_1^2 + \lambda^2 \tau_2^2)^{(n-1)/2} \tau_1, \quad (2.25)$$

$$u_z + \varepsilon^2 v_x = 2\varepsilon \lambda (\tau_1^2 + \lambda^2 \tau_2^2)^{(n-1)/2} \tau_2. \quad (2.26)$$

Boundary conditions at the base  $z = b$  are

$$v = ub_x, \quad (2.27)$$

$$\frac{-2\frac{\varepsilon}{\lambda} b_x \tau_1 + (1 - \varepsilon^2 b_x^2) \tau_2}{\sqrt{1 + \varepsilon^2 b_x^2}} = (1 + \varepsilon^2 b_x^2)^{\frac{m}{2}} |u|^{m-1} u, \quad (2.28)$$

while at the upper surface  $z = h$ ,

$$\left(p - \frac{\varepsilon}{\lambda} \tau_1\right) h_x + \tau_2 = 0, \quad (2.29)$$

$$-p - \frac{\varepsilon}{\lambda} \tau_1 - \varepsilon^2 \tau_2 h_x = 0, \quad (2.30)$$

$$h_t + u h_x = v + a. \quad (2.31)$$

The depth-integrated form of (2.31) remains as before:

$$h_t + q_x = a, \quad q = \int_b^h u dz. \quad (2.32)$$

### 3. Blatter's Model

In this section, we analyze the higher order model due to Blatter (18), which is widely used as a glacier flow simulation tool and has been studied extensively from the perspective of numerical analysis (33, 34, 35, 36, 37). The main feature of Blatter's model is that

the vertical component of total stress,  $p + \tau_1$  in dimensional terms, is approximated as hydrostatic. This is then used as a closure relation for  $p$ , eliminating the pressure variable  $p$  from the problem. Along with a further simplification of the constitutive relation (2.6), treating  $v_x$  as small compared with  $u_z$ , this leads to a closed model for the horizontal velocity  $u$  alone, reducing the elliptic system of partial differential equations (2.1)–(2.10) for  $u$  and  $v$  to a scalar elliptic problem for  $u$  only, from which the vertical velocity component  $v$  can be computed *a posteriori* from (2.3) through a simple quadrature. This reduction leads to significant computational gains while retaining a representation of both, normal and shear stresses in force balance as well as a representation of sliding and vertical shearing in the computation of fluid flux (24). In addition, an extensive literature demonstrates the well-posedness and convergence of discretized versions of Blatter’s model (33, 34, 35, 36, 37). These features account for the popularity of Blatter’s model as a numerical simulation tool.

Our aim here is to show that the approximations underlying Blatter’s model hold true for a thin film regardless of whether sliding is slow or rapid. More specifically, we will show that the error incurred in the horizontal velocity field  $u$  in making these approximations can be estimated asymptotically as being of  $O(\varepsilon^2)$  regardless of the value of the slip-related stress parameter  $\lambda$ . The reason why we focus on the error in  $u$  is that  $u$  is the primary variable needed to describe the dynamics of the thin film (as opposed to, say, the stress components  $\tau_1$  and  $\tau_2$ ): given a velocity field to error  $O(\varepsilon^2)$ , one can then use the depth-integrated mass conservation equation (2.32) to determine the surface evolution to the same error.

We begin with a derivation of Blatter’s model that differs somewhat from the one given in Blatter’s original paper. As indicated in appendix A, the derivation given in (18) is not self-consistent: the scalings he chooses lead to a dimensionless model which, if truncated at leading order in his aspect ratio parameter, has no solution. Moreover, his original paper does not consider the effect of different sliding regimes. We use these two observations as motivation for revisiting the construction of the model and for attempting to estimate the error that is incurred in using the model.

### 3.1 The construction of Blatter’s model

Blatter’s model arises if we simply drop terms of  $O(\varepsilon^2)$  in (2.22)–(2.31), paying no attention for the time being to complications that may arise when  $\lambda$  is small or large. Force balance in the fluid and vanishing traction at the upper surface, equations (2.22)–(2.23) and (2.29)–(2.30) then become

$$\frac{\varepsilon}{\lambda} \tau_{1,x} + \tau_{2,z} - p_x = 0 \quad \text{for } b < z < h, \quad (3.1)$$

$$-\frac{\varepsilon}{\lambda} \tau_{1,z} - p_z - 1 = O(\varepsilon^2) \quad \text{for } b < z < h, \quad (3.2)$$

$$\left(p - \frac{\varepsilon}{\lambda} \tau_1\right) h_x + \tau_2 = 0 \quad \text{on } z = h, \quad (3.3)$$

$$-p - \frac{\varepsilon}{\lambda} \tau_1 = O(\varepsilon^2) \quad \text{on } z = h. \quad (3.4)$$

From (3.2) and (3.4) we obtain as promised that the  $zz$ -component of stress is hydrostatic to an error of  $O(\varepsilon^2)$ :

$$p + \frac{\varepsilon}{\lambda} \tau_1 = (h - z) + O(\varepsilon^2). \quad (3.5)$$

Substituting for  $p$  in (3.1), this yields

$$2\frac{\varepsilon}{\lambda}\tau_{1,x} + \tau_{2,z} - h_x = O(\varepsilon^2). \quad (3.6)$$

Meanwhile, for  $b < z < h$ , dropping terms of  $O(\varepsilon^2)$  in (2.24)–(2.26) leads to

$$u_x + v_z = 0, \quad (3.7)$$

$$u_x = (\tau_1^2 + \lambda^2\tau_2^2)^{(n-1)/2} \tau_1, \quad (3.8)$$

$$u_z = 2\varepsilon\lambda(\tau_1^2 + \lambda^2\tau_2^2)^{(n-1)/2} \tau_2 + O(\varepsilon^2), \quad (3.9)$$

while, from (2.27)–(2.28), we have at the base  $z = b$

$$v = ub_x, \quad (3.10)$$

$$-2\frac{\varepsilon}{\lambda}b_x\tau_1 + \tau_2 = |u|^{m-1}u + O(\varepsilon^2). \quad (3.11)$$

Again, we substitute for  $p$  in (3.3) to find

$$-2\frac{\varepsilon}{\lambda}h_x\tau_1 + \tau_2 = O(\varepsilon^2) \quad (3.12)$$

on  $z = h$ , while (2.31) remains unchanged as

$$h_t + uh_x = v + a. \quad (3.13)$$

Equations (3.6), (3.8), (3.9), (3.11) and (3.12) with the  $O(\varepsilon^2)$  corrections omitted constitute Blatter's higher-order model for horizontal velocity  $u$  with a power-law sliding law at the base of the fluid film. The well-posedness of this model has been studied in numerous papers (**33**, **34**, **35**, **36**, **37**). Once  $u$  has been determined, equations (3.7) and (3.10) can be integrated easily to find the vertical velocity component  $v$ , while  $p$  can also be computed *a posteriori* from (3.5). The evolution of the upper free surface still satisfies (2.32), whose solution requires that  $u$  be determined in order to find flux  $q$ .

The derivation above has proceeded without regard to the value of  $\lambda$ . In fact, it is most obviously appropriate when  $\lambda = O(1)$ . All of the terms omitted in writing (3.2)–(3.13) do not involve  $\lambda$ , but nonetheless it is not immediately clear that the error involved in dropping the  $O(\varepsilon^2)$  terms above remains of  $O(\varepsilon^2)$  when  $\lambda$  is large or small.

To ensure that the model remains valid to  $O(\varepsilon^2)$  regardless of the behaviour of  $\lambda$  thus requires a more detailed analysis. Our approach in what follows is to show that the solution to Blatter's model has the same asymptotic expansion for the horizontal velocity  $u$  to an error of  $O(\varepsilon^2)$  as the full Stokes flow problem (2.22)–(2.30). More specifically, we show that the omitted  $O(\varepsilon^2)$  terms in (3.6)–(3.12) do not enter into the determination of terms in the expansion of  $u$  below  $O(\varepsilon^2)$ .

The appropriate asymptotic expansions do depend on the behaviour of  $\lambda$ , but in all cases, we find that the omitted  $O(\varepsilon^2)$  terms do not affect the expansion of  $u$  to an error of  $O(\varepsilon^2)$ . In order to show this, we are forced to consider three distinct parameter regimes: firstly, an intermediate regime with  $\lambda = O(1)$ , in which sliding is rapid but normal stresses play a higher-order role in force balance, secondly, a fast sliding regime with  $\lambda \ll 1$  in which

normal stresses and shear stresses both play a leading order role in force balance in the manner of a membrane-like flow, and thirdly, a slow sliding regime with  $\lambda \gg 1$  that yields a classical lubrication approximation at leading order.

A word on the level of detail that we present below is appropriate before we start. In order to keep the paper to a reasonable length, we do not provide explicit solutions for all higher order terms in the expansions, but merely show how they can be calculated once lower order solutions have been found. The aim here is not to construct asymptotic expansions as a practical means of approximating solutions, but rather to investigate the asymptotic structure of the flow problem and to show that Blatter's model and the underlying Stokes equations are equivalent as models for  $u$  to  $O(\varepsilon^2)$ .

### 3.2 Intermediate sliding: the case $\lambda = O(1)$

This is the easiest case to deal with, as the derivation of Blatter's model is most obviously correct in this parametric regime. With  $\lambda = O(1)$ , we can expand the dependent variables in asymptotic expansions in powers of  $\varepsilon$  up to an error of  $O(\varepsilon^2)$ :

$$\begin{aligned} u &\sim u^{(0)} + \varepsilon u^{(1)} + O(\varepsilon^2), & v &\sim v^{(0)} + \varepsilon v^{(1)} + O(\varepsilon^2), \\ \tau_1 &\sim \tau_1^{(0)} + \varepsilon \tau_1^{(1)} + O(\varepsilon^2), & \tau_2 &\sim \tau_2^{(0)} + \varepsilon \tau_2^{(1)} + O(\varepsilon^2), & p &\sim p^{(0)} + \varepsilon p^{(1)} + O(\varepsilon^2). \end{aligned} \quad (3.14)$$

At leading order, we obtain the following from (3.5)–(3.13). In  $b < z < h$ , we have

$$p^{(0)} = (h - z), \quad (3.15)$$

$$\tau_{2,z}^{(0)} - h_x = 0, \quad (3.16)$$

$$u_x^{(0)} + v_z^{(0)} = 0, \quad (3.17)$$

$$u_x^{(0)} = \left( \tau_1^{(0)2} + \lambda^2 \tau_2^{(0)2} \right)^{(n-1)/2} \tau_1^{(0)}, \quad (3.18)$$

$$u_z^{(0)} = 0, \quad (3.19)$$

while at the base  $z = b$

$$v^{(0)} = u^{(0)} b_x, \quad (3.20)$$

$$\tau_2^{(0)} = |u^{(0)}|^{m-1} u^{(0)}, \quad (3.21)$$

and at the upper surface  $z = h$  we have

$$\tau_2^{(0)} = 0. \quad (3.22)$$

Hence we have the simple solution

$$\begin{aligned} p^{(0)} &= (h - z), & \tau_2^{(0)} &= -(h - z) h_x, \\ u^{(0)} &= -(h - b)^{1/m} |h_x|^{1/m-1} h_x, & v^{(0)} &= u^{(0)} b_x - u_x^{(0)} (z - b). \end{aligned} \quad (3.23)$$

To  $O(\varepsilon)$ , pressure is hydrostatic, vertical shear stress balances the driving stress, and the velocity field is a plug flow determined by a local balance between friction at the bed and

driving stress. Lastly, normal stress can be found from the algebraic equation (3.18), in which  $u_x^{(0)}$  and  $\tau_2^{(0)}$  can be substituted from above.

Given the leading order solution,  $O(\varepsilon)$  corrections can be computed by expanding (3.5)–(3.12): in  $b < z < h$ ,

$$p^{(1)} = -\frac{1}{\lambda}\tau_1^{(0)}, \quad (3.24)$$

$$\frac{2}{\lambda}\tau_{1,x}^{(0)} + \tau_{2,z}^{(1)} = 0, \quad (3.25)$$

$$u_x^{(1)} + v_z^{(1)} = 0, \quad (3.26)$$

$$\begin{aligned} u_x^{(1)} &= n \left( \tau_1^{(0)2} + \lambda^2 \tau_2^{(0)2} \right)^{(n-1)/2} \tau_1^{(1)} \\ &\quad + (n-1)\lambda^2 \left( \tau_1^{(0)2} + \lambda^2 \tau_2^{(0)2} \right)^{\frac{n-3}{2}} \tau_2^{(0)} \left( \tau_2^{(0)} - \tau_1^{(0)} \right) \tau_2^{(1)} \end{aligned} \quad (3.27)$$

$$u_z^{(1)} = 2\lambda \left( \tau_1^{(0)2} + \lambda^2 \tau_2^{(0)2} \right)^{(n-1)/2} \tau_2^{(0)}, \quad (3.28)$$

while at the base  $z = b$

$$v^{(1)} = u^{(1)}b_x, \quad (3.29)$$

$$-\frac{2}{\lambda}b_x\tau_1^{(0)} + \tau_2^{(1)} = m|u^{(0)}|^{m-1}u^{(1)}, \quad (3.30)$$

and at the upper surface  $z = h$  we have

$$-\frac{2}{\lambda}\tau_1^{(0)}h_x + \tau_2^{(1)} = 0, \quad (3.31)$$

We do not solve this problem explicitly here. It is clear that the first order problem is linear in  $u^{(1)}$ ,  $v^{(1)}$ ,  $\tau_1^{(1)}$ ,  $\tau_2^{(1)}$  and  $p^{(1)}$ , and its solution can be computed through simple quadratures once the zeroth order solution is known: (3.24) determines  $p^{(1)}$  while (3.25) and (3.31) allow us to compute  $\tau_2^{(1)}$ , from which  $u^{(1)}$  at  $z = b$  can be found through (3.30). The form of  $u^{(1)}$  above  $z = b$  then follows from (3.28), and  $v^{(1)}$  can be found from (3.29) and (3.26).

We observe that  $u^{(1)}$ ,  $\tau_1^{(1)}$  and  $\tau_2^{(1)}$  can be computed from (3.25), (3.27), (3.28), (3.30) and (3.31) alone, and none of these involve the vertical velocity component  $v$  or pressure  $p$ . The elimination of the vertical velocity component  $v$  and pressure  $p$  from Blatter's model (equations (3.6), (3.8), (3.9), (3.11) and (3.12)) imply that the expansions for  $v$  and  $p$  can be computed *a posteriori* and are not required in the calculation of  $\tau_1$ ,  $\tau_2$  and  $u$ . This will be a feature throughout our asymptotic analysis: our focus will generally be on expansions for  $\tau_1$ ,  $\tau_2$ , and  $u$ , as  $v$  can generally be computed from the incompressibility condition (3.7) along with the velocity boundary condition (3.10) once the velocity field  $u$  has been determined to sufficient accuracy, while  $p$  can be determined through (3.5) once  $\tau_2$  is known.

As promised, we have seen that  $u$  can be computed to  $O(\varepsilon^2)$  from Blatter's model when  $\lambda = O(1)$ , and only  $O(\varepsilon^2)$  corrections in the expansions would require terms going beyond those retained in (3.5)–(3.12). To an error of  $O(\varepsilon^2)$ , (3.6)–(3.13) and (2.22)–(2.31) have the same asymptotic structure. It remains to show the same for  $\lambda \ll 1$  and  $\lambda \gg 1$ .

Before we do so, we note a feature of the expansion above that will be important in section 4, where we consider higher order models that are simplified even further than Blatter's model. In order to compute  $u$  to an error of  $O(\varepsilon^2)$ , we only need to know  $\tau_1$  to an error of  $O(\varepsilon)$ : although  $\tau_1^{(1)}$  can be computed from (3.27), we do not need to know its value in order to compute  $u^{(1)}$  from (3.28).

### 3.3 Fast sliding (i): the case $\varepsilon \lesssim \lambda \ll 1$

We now consider the fast sliding case  $\varepsilon \lesssim \lambda \ll 1$ , which corresponds physically to normal stresses playing a more significant role in force balance than considered above; in particular, they appear at leading order when  $\varepsilon \sim \lambda$ . We proceed as in the previous section by expanding in an asymptotic expansion up to error  $O(\varepsilon^2)$ . This is complicated slightly by the fact that we need to expand in both  $\lambda$  and  $\varepsilon$ , but as we will show below, it is possible to do this in the relatively simple form

$$\begin{aligned}\tau_1 &\sim \tau_1^{(0)} + \left(\lambda^2 \tau_1^{(1)} + \lambda^4 \tau_1^{(2)} + \dots\right) + O(\varepsilon\lambda), \\ \tau_2 &\sim \tau_2^{(0)} + \varepsilon\lambda \left(\tau_2^{(1)} + \lambda^2 \tau_2^{(2)} + \lambda^4 \tau_2^{(3)} + \dots\right) + O(\varepsilon^2), \\ p &\sim p^{(0)} + \varepsilon\lambda \left(p^{(1)} + \lambda^2 p^{(2)} + \lambda^4 p^{(3)} + \dots\right) + O(\varepsilon^2), \\ u &\sim u^{(0)} + \varepsilon\lambda \left(u^{(1)} + \lambda^2 u^{(2)} + \lambda^4 u^{(3)} + \dots\right) + O(\varepsilon^2), \\ v &\sim v^{(0)} + \varepsilon\lambda \left(v^{(1)} + \lambda^2 v^{(2)} + \lambda^4 v^{(3)} + \dots\right) + O(\varepsilon^2).\end{aligned}\tag{3.32}$$

Here, the expansions in powers of  $\lambda^2$  are to be truncated so that the error in the expansion for  $\tau_1$  is  $O(\varepsilon\lambda)$  as indicated, while for the remaining variables, the error is  $O(\varepsilon^2)$ . In other words, the highest order terms in the expansions of  $\tau_2$ ,  $p$ ,  $u$  and  $v$  are  $\tau_2^{(N)}$ ,  $p^{(N)}$ ,  $u^{(N)}$  and  $v^{(N)}$  where  $N$  is such that  $\varepsilon\lambda^{1+2(N-1)} \gg \varepsilon^2$  but  $\varepsilon\lambda^{1+2N} \lesssim \varepsilon^2$ . Meanwhile the highest order correction term in the expansion of  $\tau_1$  is  $\tau_1^{(N')}$ , where  $N'$  is the natural number for which  $\lambda^{2N'} \gg \varepsilon\lambda$ , but  $\lambda^{2(N'+1)} \lesssim \varepsilon\lambda$ . Hence  $N' = N$  and the expansions for  $\tau_1$ ,  $\tau_2$ ,  $p$ ,  $u$  and  $v$  are all developed to the same order.

Substituting (3.32) in (3.6), (3.8), (3.9), (3.11) and (3.12) and expanding the right-hand sides of (3.8) and (3.9) in Taylor series, we have the following at leading order:

$$2\frac{\varepsilon}{\lambda}\tau_{1,x}^{(0)} + \tau_{2,z}^{(0)} - h_x = 0,\tag{3.33}$$

$$\tau_1^{(0)} = |u_x^{(0)}|^{1/n-1}u_x^{(0)},\tag{3.34}$$

$$u_z^{(0)} = 0,\tag{3.35}$$

for  $b < z < h$ , and

$$-2\frac{\varepsilon}{\lambda}b_x\tau_1^{(0)} + \tau_2^{(0)} = |u^{(0)}|^{m-1}u^{(0)} \quad \text{on } z = b,\tag{3.36}$$

$$-2\frac{\varepsilon}{\lambda}\tau_1^{(0)}h_x + \tau_2^{(0)} = 0 \quad \text{on } z = h.\tag{3.37}$$

We see immediately that the *ansatz* in (3.32) forces terms of  $\varepsilon/\lambda$  to be retained at leading

order in the expansion of  $\tau_2$ ,  $p$ ,  $u$  and  $v$  even though these can be small for the parameter regime  $\varepsilon \lesssim \lambda \ll 1$  considered here. This approach is perfectly legitimate, however, as the retained  $\varepsilon/\lambda$  terms are much larger than the next order of correction in (3.32), which is of  $O(\varepsilon\lambda)$ . Hence retaining  $O(\varepsilon/\lambda)$  terms at leading order does not affect the ordering in the expansion.

From (3.35),  $u \sim u^{(0)}(x, t)$  is independent of  $z$  to  $O(\varepsilon\lambda)$ , and from (3.34),  $\tau_1 \sim \tau_1^{(0)}(x, t)$  is also independent of  $z$  to  $O(\lambda^2)$ . Integrating (3.33) from  $z = b$  to  $z = h$  then yields

$$2\frac{\varepsilon}{\lambda}(h-b)\tau_{1,x}^{(0)} + \tau_2^{(0)}|_{z=h} - \tau_2^{(0)}|_{z=b} - (h-b)h_x = 0. \quad (3.38)$$

Substituting for  $\tau_2^{(0)}$  using the boundary conditions (3.36) and (3.37) and using the constitutive relation (3.34) for  $\tau_1^{(0)}$  gives, after a few manipulations,

$$2\frac{\varepsilon}{\lambda} \left[ (h-b)|u_x^{(0)}|^{1/n-1}u_x^{(0)} \right]_x - |u^{(0)}|^{m-1}u^{(0)} - (h-b)h_x = 0. \quad (3.39)$$

This is the one-dimensional version of the membrane-like ice flow model of **(13)**. Specifically, (3.39), is a second-order elliptic equation from which the leading order velocity term  $u^{(0)}$  can be computed. Given  $u^{(0)}$ ,  $\tau_1^{(0)}$  can then be calculated from (3.34) by differentiating, and  $\tau_2^{(0)}$  follows from (3.33) and (3.37) through a simple integration. As before, the calculation of  $p^{(0)}$  and  $v^{(0)}$  decouples from that of  $u^{(0)}$ : from (3.5), we find that  $p^{(0)} = h - z - (\varepsilon/\lambda)\tau_1^{(0)}$ , while from (3.7) and (3.10),  $v^{(0)} = u^{(0)}b_x - u_x^{(0)}(z - b)$ .

If  $\lambda \sim \varepsilon$ , the leading order solution is all we need, as all terms omitted in (3.33)–(3.37) are then of  $O(\varepsilon^2)$ . In terms of the truncation level  $N$  in the expansion (3.32),  $\lambda \sim \varepsilon$  implies  $N = 0$ . This distinguished limit in fact underlies many of the analyses of thin film flows with extensional stresses **(4, 5, 38)**; the novelty here is therefore in part that we go beyond the case  $\lambda \sim \varepsilon$ .

If therefore  $\varepsilon \ll \lambda$ , we find the following generic problem at  $r$ -th order, where  $r \leq N$ . In  $b < z < h$ , we have

$$p^{(r)} = -\tau_1^{(r)}, \quad (3.40)$$

$$2\tau_{1,x}^{(r)} + \tau_{2,z}^{(r)} = 0, \quad (3.41)$$

$$u_x^{(r)} + v_z^{(r)} = 0, \quad (3.42)$$

$$0 = \alpha_r \left( \tau_1^{(0)}, \tau_1^{(1)}, \dots, \tau_1^{(r)}, \tau_2^{(0)} \right), \quad (3.43)$$

$$u_z^{(r)} = \beta_r \left( \tau_1^{(0)}, \tau_1^{(1)}, \dots, \tau_1^{(r-1)}, \tau_2^{(0)} \right) \quad (3.44)$$

where  $\alpha_r$  and  $\beta_r$  are functions that arise from the expansion of the right-hand sides of (3.8) and (3.9) in Taylor series. As boundary conditions, we obtain from (3.10), (3.11) and (3.12)

$$v^{(r)} = u^{(r)}b_x \quad \text{on } z = b, \quad (3.45)$$

$$-2\tau_1^{(r-1)}b_x + \tau_2^{(r)} = m|u^{(0)}|^{m-1}u^{(r)} \quad \text{on } z = b, \quad (3.46)$$

$$-2\tau_1^{(r-1)}h_x + \tau_2^{(r)} = 0 \quad \text{on } z = h. \quad (3.47)$$

These relations are easy to reconstruct for the first order correction  $r = 1$ , where

$$\alpha_1 \left( \tau_1^{(0)}, \tau_1^{(1)}, \tau_2^{(0)} \right) = |\tau_1^{(0)}|^{n-1} \tau_1^{(1)} + \frac{n-1}{2} |\tau_1^{(0)}|^{n-3} \tau_1^{(0)} \left( 2\tau_1^{(0)} \tau_1^{(1)} + \tau_2^{(0)2} \right), \quad (3.48)$$

$$\beta_1 \left( \tau_1^{(0)}, \tau_2^{(0)} \right) = |\tau_1^{(0)}|^{n-1} \tau_2^{(0)}. \quad (3.49)$$

For more general  $r$ , the form of the functions  $\alpha_r$  and  $\beta_r$  is derived in appendix B.1, where we show that  $\alpha_r$  is always affine in  $\tau_1^{(r)}$ , and that equation (3.43) can generally be inverted uniquely for  $\tau_1^{(r)}$  as a function of  $\tau_1^{(0)}, \dots, \tau_1^{(r-1)}$  and  $\tau_2^{(0)}$ . The correction problem consisting of (3.41)–(3.47) and (3.43)–(3.44) is then clearly solvable. Once lower-order corrections to  $\tau_1$  have been computed, (3.43) implicitly determines  $\tau_1^{(r)}$ . Once  $\tau_1^{(r)}$  is known,  $\tau_2^{(r)}$  can be calculated from (3.41) and (3.47). This finally allows  $u^{(r)}$  to be computed through (3.46) and (3.44).  $p^{(r)}$  and  $v^{(r)}$  can be computed from (3.40) and from (3.42) and (3.45) once  $\tau_1^{(r)}$  and  $u^{(r)}$  have been determined, and their calculation again decouples from the rest of the problem.

As we can solve in this way for higher order corrections up to the truncation level  $N$ , it follows that Blatter's model allows an asymptotic expansion to be developed up to an error of  $O(\varepsilon^2)$  in  $u$ , and the omitted  $O(\varepsilon^2)$  terms in (3.6)–(3.12) only enter into the determination of  $u$  at  $O(\varepsilon^2)$  and higher.

This completes our task for the parameter regime considered here. That said, there are several observations to make before we move on. Retaining  $O(\varepsilon/\lambda)$  terms at leading order was essential in order to develop the simple asymptotic expansions in (3.32); otherwise, expansion in powers of  $\varepsilon/\lambda$  would have become necessary, and the relative ordering of powers of  $\lambda$  and of  $\varepsilon/\lambda$  would have had to be considered in constructing higher order corrections. Nonetheless, we can consider the leading order solution in the limit  $\varepsilon/\lambda \ll 1$ . From (3.38), (3.39) and (3.34), we have

$$\begin{aligned} u^{(0)} &\sim -(h-b)^{1/m} |h_x|^{1/m-1} h_x + O(\varepsilon/\lambda), & \tau_2^{(0)} &= -h_x(h-z) + O(\varepsilon/\lambda), \\ \tau_1^{(0)} &= -(h-b)^{1/(mn)} |h_x|^{(1-mn)/(mn)} h_x + O(\varepsilon/\lambda). \end{aligned} \quad (3.50)$$

We observe that this solution for  $u^{(0)}$  and  $\tau_2^{(0)}$  is the same to  $O(\varepsilon/\lambda)$  as the leading order solution in (3.23) in the previous subsection. The leading order model (3.39) constructed above, which retains the  $O(\varepsilon/\lambda)$  correction, can therefore be thought of as an improvement over the leading order model (3.16)–(3.22) when  $\varepsilon \ll \lambda \ll 1$ .

Conversely, in the limit  $\lambda \sim \varepsilon$ , it is clear that Blatter's model retains some terms of  $O(\varepsilon^2)$  (namely terms including factors of the form  $\varepsilon\lambda$  and  $\lambda^2$ ) while dropping others. This is the price that must be paid for having a single simplified model that is valid for all values of  $\lambda$ : the  $O(\lambda^2)$  and  $O(\varepsilon\lambda)$  terms do feature at lower order than  $O(\varepsilon^2)$  when  $\lambda \gg \varepsilon$ . Specifically, we have seen that the leading order model (3.39), which ignores terms of  $O(\lambda^2)$  and  $O(\varepsilon\lambda)$ , allows  $u$  to be calculated only to an error of  $O(\varepsilon\lambda)$ , which is worse than the  $O(\varepsilon^2)$  error in Blatter's model when  $\varepsilon \ll \lambda \ll 1$ . Lastly, with a view to constructing other simplified higher order models in section 4, we note that, in the expansion of Blatter's model above,  $\tau_1$  only needs to be computed to an error of  $O(\varepsilon\lambda)$  in order to compute  $u$  to an error of  $O(\varepsilon^2)$ .

### 3.4 Fast sliding (ii): the case $\lambda \ll \varepsilon$

This parameter regime corresponds physically to normal stresses dominating over friction in the balance of forces on the fluid. In effect, we have a ‘free film’ as discussed for a Newtonian fluid in the context of surface tension effects in Erneux & Davis (3). With  $\lambda \ll \varepsilon$ , the factors  $\varepsilon/\lambda$  appearing in equations (3.6), (3.2), (3.11), (3.12) and (3.4) are then large, so it is no longer clear whether Blatter’s model applies. To resolve this, a rescaling is required.

In fact, (3.6), (3.11) and (3.12) suggest  $\tau_1 \sim O(\lambda/\varepsilon)$ , which indicates that  $\tau_1$  is no longer scaled appropriately, and from (3.11), neither is  $u$ . Physically, the reason for this is that  $\lambda \ll \varepsilon$  corresponds to very weak friction at the base of the fluid, in which case the size of normal stresses  $\tau_1$  is controlled by a balance between the normal stress gradient  $\tau_{1,x}$  and driving stress  $-\rho g(h-b)h_x$ , that is, through the balance of scales

$$[\tau_1]/[x] = \rho g[h]^2/[x], \quad (3.51)$$

while (2.14), which assumes that velocity is controlled by friction rather than normal stress, is no longer appropriate.

Re-introducing the asterisks for now to avoid ambiguity, the appropriate re-scaling is

$$\begin{aligned} \tau_1^{**} &= \varepsilon \lambda^{-1} \tau_1^*, & \tau_2^{**} &= \tau_2^*, & p^{**} &= p^*, \\ u^{**} &= \varepsilon^n \lambda^{-n} u^*, & v^{**} &= \varepsilon^n \lambda^{-n} v^*, & a^{**} &= \varepsilon^n \lambda^{-n} a^*, & t^{**} &= \lambda^n \varepsilon^{-n} t^*. \end{aligned} \quad (3.52)$$

We also define a friction parameter  $\nu$  through

$$\nu = \lambda^{mn} \varepsilon^{-mn}, \quad (3.53)$$

where we see immediately that  $\nu \ll 1$  when  $\lambda \ll \varepsilon$ . The Stokes flow equations (2.22)–(2.31) then become the following, once again dispensing with the double asterisks. In the fluid flow domain  $b < z < h$ , we have

$$\tau_{1,x} + \tau_{2,z} - p_x = 0, \quad (3.54)$$

$$-\tau_{1,z} + \varepsilon^2 \tau_{2,x} - p_z - 1 = 0, \quad (3.55)$$

$$u_x + v_z = 0, \quad (3.56)$$

$$u_x = (\tau_1^2 + \varepsilon^2 \tau_2^2)^{(n-1)/2} \tau_1, \quad (3.57)$$

$$u_z + \varepsilon^2 v_x = 2\varepsilon^2 (\tau_1^2 + \varepsilon^2 \tau_2^2)^{(n-1)/2} \tau_2, \quad (3.58)$$

while boundary conditions at the base  $z = b$  are

$$v = ub_x, \quad (3.59)$$

$$\frac{-2b_x \tau_1 + (1 - \varepsilon^2 b_x^2) \tau_2}{\sqrt{1 + \varepsilon^2 b_x^2}} = \nu (1 + \varepsilon^2 b_x^2)^{\frac{m}{2}} |u|^{m-1} u, \quad (3.60)$$

and at the upper surface  $z = h$  we have

$$(p - \tau_1) h_x + \tau_2 = 0, \quad (3.61)$$

$$-p - \tau_1 - \varepsilon^2 \tau_2 h_x = 0, \quad (3.62)$$

$$h_t + u h_x = v + a. \quad (3.63)$$

In other words, we retrieve the same equations as (2.22)–(2.31), but with  $\lambda = \varepsilon$ , and a small parameter  $\nu$  appended to the right-hand side of (2.28). This small parameter signifies that, for  $O(1)$  velocities, shear stresses at the base are low.

We can now obtain Blatter’s model (3.5)–(3.12) if we drop all terms of  $O(\varepsilon^2)$  except those on the right-hand side of (3.57) and (3.58), and again eliminate the pressure variable  $p$  as in (3.5):

$$p + \tau_1 = (h - z) + O(\varepsilon^2), \quad (3.64)$$

$$2\tau_{1,x} + \tau_{2,z} - h_x = O(\varepsilon^2), \quad (3.65)$$

$$u_x + v_z = 0, \quad (3.66)$$

$$u_x = (\tau_1^2 + \varepsilon^2 \tau_2^2)^{(n-1)/2} \tau_1, \quad (3.67)$$

$$u_z = 2\varepsilon^2 (\tau_1^2 + \varepsilon^2 \tau_2^2)^{(n-1)/2} \tau_2 + O(\varepsilon^2) \quad (3.68)$$

in  $b < z < h$ , while

$$-2b_x \tau_1 + \tau_2 = \nu |u|^{m-1} u + O(\varepsilon^2), \quad (3.69)$$

$$v = ub_x \quad (3.70)$$

at  $z = b$  and

$$-2\tau_1 h_x + \tau_2 = 0 \quad (3.71)$$

at  $z = h$ .

In arriving at Blatter’s model for this parameter regime, we again have to retain some higher-order terms (those on the right-hand sides of (3.67) and (3.68)) selectively while dropping others that appear at the same order. As before, this is the price to be paid for having a single model capable of describing all sliding regimes.

With Blatter’s model in the form (3.64)–(3.71), it is a simple exercise to show that the Stokes equations (3.54)–(3.62) have the same solution to an error of  $O(\varepsilon^2)$ : if we write  $u = u^{(0)} + O(\varepsilon^2)$ ,  $\tau_1 = \tau_1^{(0)} + O(\varepsilon^2)$ ,  $\tau_2 = \tau_2^{(0)} + O(\varepsilon^2)$  etc., then we recover equations mirroring (3.33)–(3.37), and hence the viscous membrane model (3.39). In other words, we have

$$2(h|u_x^{(0)}|^{1/n-1} u_x^{(0)})_x - \nu |u^{(0)}|^{m-1} u^{(0)} - (h-b)h_x = 0, \quad (3.72)$$

or the same equation with the term  $\nu |u^{(0)}|^{m-1} u^{(0)}$  omitted if  $\nu \lesssim \varepsilon^2$  (formally, this is the limit in which friction does not affect the velocity field above an error of  $O(\varepsilon^2)$  and may therefore be omitted: this is the free film limit of Erneux and Davis (**3**)). Importantly, we obtain the same solution  $u^{(0)}$  to an error of  $O(\varepsilon^2)$  from Blatter’s model as from the original Stokes model (2.22)–(2.30), as well as from MacAyeal’s model (3.72). As required, Blatter’s is again accurate in  $u$  to  $O(\varepsilon^2)$ .

### 3.5 Slow sliding (i): the case $1 \ll \lambda \lesssim \varepsilon^{-1/n}$

Finally, we turn to the case of slow sliding, for which  $\lambda \gg 1$ . Physically, this is the case in which vertical shearing in the fluid plays a significant role in generating flux  $q$ . The leading order terms in a perturbation expansion for this case reproduce the standard lubrication models of (**2**, **9**, **10**, **11**), as we shall see below. Nonetheless, slow sliding turns out to be the most complicated of the parameter regimes we have to consider in this paper.

Specifically, the shear-thinning power law rheology of the fluid significantly complicates the derivation of higher order terms. In a standard lubrication approximation for a thin film flow with a free upper surface, stress vanishes at that surface. With a shear-thinning power law rheology, this however corresponds to infinite viscosity, and the standard lubrication approximation must break down near the upper surface, where a high-viscosity boundary layer with significant normal stresses exists. This boundary layer, first studied for  $n = 3$  and with slow slip at the base of the fluid ( $\lambda \sim \varepsilon^{-1/n}$ ) by Johnson & McMeeking (28) (section 3), makes our objective of studying higher order expansions up to  $O(\varepsilon^2)$  considerably harder than in the previously considered parameter regimes (in fact, as we shall see at the end of section 3.5.3, even Johnson & McMeeking do not carry out their expansion to an error of  $O(\varepsilon^2)$ ).

In the development that follows, we also apply matched asymptotics to the high-viscosity boundary layer problem, albeit to higher order and for more general parameter choices than (28). In the next subsection, we study the outer problem for fluid flow outside the boundary layer at the upper surface, followed by the inner problem describing the boundary layer in §3.5.2, and in §3.5.3 we consider asymptotic matching between the two solutions that arise.

Recall that our main objective is to show that Blatter's model and the original Stokes equations lead to the same perturbation expansions up to an error in  $u$  of  $O(\varepsilon^2)$ . In sections 3.2–3.3, we were able to use Blatter's model (3.6)–(3.12) alone to achieve this, as the terms omitted in Blatter's model were known to be of  $O(\varepsilon^2)$ . A subtle complication arises, however, when we have to consider a boundary layer at the upper surface. In this boundary layer, a rescaling becomes necessary, and it is not immediately clear that the terms omitted in Blatter's model remain of  $O(\varepsilon^2)$  under this rescaling. This affects not only the expansion of the solution in the boundary layer, but also the closure relation (3.5) for pressure, which relies on the boundary condition (3.4) at the upper surface. As this closure relation was used to derive the force balance equation (3.6), which also appears in the outer problem for Blatter's model, the question arises as to whether the  $O(\varepsilon^2)$  term in (3.4) remains of  $O(\varepsilon^2)$  under the necessary rescaling in the boundary layer: if it does not, then the error in (3.5) and hence in (3.6) may no longer be of  $O(\varepsilon^2)$  as indicated.

To avoid this problem, we do not use the integrated form (3.5) of (3.2) and (3.4), but base our analysis below on equations (3.1)–(3.4) and (3.7)–(3.11). We will show that the terms omitted in these equations remain of  $O(\varepsilon^2)$  in the inner region as well as the outer region, and that one can once again compute  $u$  to an error of  $O(\varepsilon^2)$  from these equations. As Blatter's model (3.5)–(3.12) is derived from (3.1)–(3.4) and (3.7)–(3.11), it then follows that Blatter's model also allows  $u$  to be computed to an error of  $O(\varepsilon^2)$ .

### 3.5.1 The outer problem

As we will see in subsection 3.5.2, the boundary layer thickness is of  $O(\lambda^{-1})$ , so we take the region defined by  $h - b > h - z \gg \lambda^{-1}$  as our outer domain, in which the shear stress  $\tau_2$  determines viscosity at leading order. As in section 3.3, finding an asymptotic solution of (3.6)–(3.12) in this outer region to an  $O(\varepsilon^2)$  error requires expansions in both,  $\varepsilon$  and  $\lambda$ .

We consider expansions in the outer domain of the form

$$\begin{aligned}
\tau_1 &\sim \lambda^{1-n} \left( \tau_1^{(0)} + \lambda^{-2n} \tau_1^{(1)} + \lambda^{-4n} \tau_1^{(2)} + \dots \right) + O(\varepsilon\lambda) \\
\tau_2 &\sim \tau_2^{(0)} + \varepsilon\lambda^{-2} \tau_2^{(i,1)} + \varepsilon\lambda^{-n} \left( \tau_2^{(1)} + \lambda^{-2n} \tau_2^{(2)} + \lambda^{-4n} \tau_2^{(3)} + \dots \right) + O(\varepsilon^2), \\
p &\sim p^{(0)} + \varepsilon\lambda^{-n} \left( p^{(1)} + \lambda^{-2n} p^{(2)} + \lambda^{-4n} p^{(3)} + \dots \right) + O(\varepsilon^2), \\
u &\sim u^{(0)} + \varepsilon\lambda^{-2} u^{(i,1)} + \varepsilon^2 \lambda^{n-2} u^{(i,2)} \\
&\quad + \varepsilon\lambda^{-n} \left( u^{(1)} + \lambda^{-2n} u^{(2)} + \lambda^{-4n} u^{(3)} + \dots \right) + O(\varepsilon^2), \\
v &\sim v^{(0)} + \varepsilon\lambda^{-n} \left( v^{(1)} + \lambda^{-2n} v^{(2)} + \lambda^{-4n} v^{(3)} + \dots \right) + O(\varepsilon^2).
\end{aligned} \tag{3.73}$$

Essentially, all the terms in this expansion except those superscripted  $(i, 1)$  and  $(i, 2)$  arise from expanding the model (3.6)–(3.12) in the outer region in powers of  $\lambda^{-2n}$  up to the error indicated (i.e., to an error of  $O(\varepsilon\lambda)$  for  $\tau_1$ , and an error of  $O(\varepsilon^2)$  for all other dependent variables.) This is analogous to, say, the expansion in powers of  $\lambda^2$  in section 3.3. To achieve an error of the desired order, these expansions in powers of  $\lambda^{-2n}$  are again to be truncated at the level indicated. Hence the highest order terms to be retained in the expansions of  $\tau_2$ ,  $p$ ,  $u$  and  $v$  are  $\tau_2^{(N)}$ ,  $p^{(N)}$ ,  $u^{(N)}$  and  $v^{(N)}$ , where  $N$  is chosen such that  $\varepsilon\lambda^{-(2N-1)n} \gg \varepsilon\lambda$  while  $\lambda^{-(2N+1)n} \lesssim \varepsilon\lambda$ . Similarly,  $\tau_1$  is expanded to up to the correction term  $\tau_1^{(N')}$  where  $\varepsilon\lambda^{-(2N'+1)n} \gg \varepsilon\lambda$  and  $\lambda^{-(2N'+3)n} \lesssim \varepsilon\lambda$ , so that  $N' = N - 1$ . In other words,  $\tau_1$  is expanded to one order lower than the other variables  $\tau_2$ ,  $p$ ,  $u$  and  $v$ .<sup>†</sup> This explains most of the structure of the proposed expansion: as we shall show later, the additional terms with superscripts  $(i, 1)$  and  $(i, 2)$  are required in addition to match with the inner (boundary layer) problem.

At leading order, we have in the outer domain from (3.1)–(3.2) and (3.7)–(3.11) that

$$\tau_{2,z}^{(0)} - p_x^{(0)} = 0, \tag{3.74}$$

$$-p_z^{(0)} - 1 = 0, \tag{3.75}$$

$$u_x^{(0)} + v_z^{(0)} = 0, \tag{3.76}$$

$$u_x^{(0)} = |\tau_2^{(0)}|^{n-1} \tau_1^{(0)}, \tag{3.77}$$

$$u_z^{(0)} = \varepsilon\lambda^n |\tau_2^{(0)}|^{n-1} \tau_2^{(0)}, \tag{3.78}$$

and at the base  $z = b$  we have from (3.10)–(3.11)

$$v^{(0)} = u^{(0)} b_x, \tag{3.79}$$

$$\tau_2^{(0)} = |u^{(0)}|^{m-1} u^{(0)}. \tag{3.80}$$

From (3.78), we see that the *ansatz* in (3.73) forces us to retain a term of  $O(\varepsilon\lambda^n)$  at leading order in the calculation of  $u$ .  $\varepsilon\lambda^n$  can be small in the parameter regime  $1 \ll \lambda \lesssim \varepsilon^{-1/n}$

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<sup>†</sup> Note that for  $\lambda \sim \varepsilon^{-1/n}$ , even the leading order term  $\lambda^{1-n} \tau_1^{(0)}$  is of the same order as the omitted term  $O(\varepsilon\lambda)$ ; correspondingly, the development below shows that  $\tau_1$  is not required at all to calculate  $u$  to  $O(\varepsilon^2)$  in the outer region for that parameter regime.

considered here, but as in section 3.3, this approach of retaining a small term is legitimate because the next order correction to  $u$  is of  $O(\varepsilon\lambda^{-n})$  or  $O(\varepsilon\lambda^{-2})$ , both of which are small compared with  $O(\varepsilon\lambda^n)$ . Hence retaining terms of  $O(\varepsilon\lambda^n)$  at leading order in  $u$  does not affect the ordering of the expansion.

Anticipating that the behaviour of the solution near the upper surface  $z = h$  will be determined through matching with a surface boundary layer, we do not impose boundary conditions there. (3.74) then has the general solution

$$\begin{aligned} p^{(0)} &= C_1^{(0)} - z, & \tau_2^{(0)} &= C_2^{(0)} + C_{1,x}^{(0)}z, \\ u^{(0)} &= \left| C_2^{(0)} + C_{1,x}^{(0)}b \right|^{1/m-1} \left[ C_2^{(0)} + C_{1,x}^{(0)}b \right] + \varepsilon\lambda^n \int_b^z |\tau_2^{(0)}(x, z')|^{n-1} \tau_2^{(0)}(x, z') dz', \\ \tau_1^{(0)} &= u_x^{(0)} / |\tau_2^{(0)}|^{n-1}, & v^{(0)} &= u^{(0)}|_{z=b} b_x - \int_b^z u_x^{(0)}(x, z') dz'. \end{aligned} \quad (3.81)$$

where  $C_1^{(0)}(x)$  and  $C_2^{(0)}(x)$  are functions of  $x$  only (we omit the time variable  $t$  from our notation here) that must be determined through asymptotic matching with the boundary layer. Note that, as indicated above, the zeroth order expression for  $u^{(0)}$  does include a term of  $O(\varepsilon\lambda^n)$ .

Next, we turn our attention to higher-order correction terms. If  $\lambda^n \sim \varepsilon^{-1}$ , then higher order terms with superscripts (1), (2) etc. in (3.73) are all of  $O(\varepsilon^2)$  or smaller and need not be considered. In other words, the truncation level for the expansions is  $N = 0$ . For  $\lambda^n \ll \varepsilon^{-1}$ , we can go on to construct these higher order corrections in the outer domain.

We consider first the expansion of (3.8) up to an order  $r \leq N' = N - 1$ . This yields equations of the form

$$0 = \gamma_r \left( \tau_1^{(0)}, \tau_1^{(1)}, \dots, \tau_1^{(r)}, \tau_2^{(0)} \right), \quad (3.82)$$

where the  $\gamma_r$  are functions that arise from expanding the right-hand side of (3.8) in a Taylor expansion. For instance, for  $r = 1$ , we have from (3.8)

$$\gamma_1(\tau_1^{(0)}, \tau_1^{(1)}, \tau_2^{(0)}) = \frac{n-1}{2} |\tau_2^{(0)}|^{n-3} \tau_1^{(0)3} + |\tau_2^{(0)}|^{n-1} \tau_1^{(1)}. \quad (3.83)$$

More generally, the form of  $\gamma_r$  is derived in appendix B.2, where we find that  $\gamma_r$  is affine in  $\tau_1^{(r)}$  and (3.83) can be inverted uniquely for  $\tau_1^{(r)}$ . Given lower order terms  $\tau_1^{(0)}, \dots, \tau_1^{(r-1)}$  and  $\tau_2^{(0)}$ ,  $\tau_1^{(r)}$  can then be determined implicitly through (3.82). In other words, the  $\tau_1^{(r)}$  terms can be found recursively up to  $\tau_1^{(N-1)}$  once the zeroth order terms  $\tau_1^{(0)}$  and  $\tau_2^{(0)}$  are known.

Next, we also expand  $u$ ,  $v$ ,  $\tau_2$  and  $p$  up to the correction terms  $u^{(N)}$ ,  $v^{(N)}$ ,  $\tau_2^{(N)}$  and  $p^{(N)}$ . The generic form of the higher order problems arising from (3.1)–(3.4), (3.7) and (3.9) at order  $r \leq N$  is then the following:

$$\tau_{1,x}^{(r-1)} + \tau_{2,z}^{(r)} - p_x^{(r)} = 0, \quad (3.84)$$

$$-\tau_{1,z}^{(r-1)} - p_z^{(r)} = 0, \quad (3.85)$$

$$u_x^{(r)} + v_z^{(r)} = 0, \quad (3.86)$$

$$u_z^{(r)} = \chi_r \left( \tau_1^{(0)}, \tau_1^{(1)}, \dots, \tau_1^{(r-1)}, \tau_2^{(0)} \right), \quad (3.87)$$

in  $b < z < h$ , where  $\chi_r$  is a function that arises from the Taylor expansion of the right-hand side of (3.9). For example, with  $r = 1$ ,

$$\chi_1(\tau_1^{(0)}, \tau_2^{(0)}) = (n-1)|\tau_2^{(0)}|^{n-3}\tau_1^{(0)2}\tau_2^{(0)}, \quad (3.88)$$

and the general form for  $\chi_r$  can be found in appendix B.2. Meanwhile, (3.10) and (3.11) yield

$$v^{(r)} = u^{(r)}b_x, \quad (3.89)$$

$$\tau_2^{(r)} = m|u^{(0)}|^{m-1}u^{(r)} \quad (3.90)$$

at  $z = b$ .

The correction problem (3.84)–(3.90) can be solved straightforwardly. Having solved for lower order terms,  $p^{(r)}$  can be determined up to a function of  $x$  only through (3.85),

$$p^{(r)} = -\tau_1^{(r-1)} + C_1^{(r)}(x), \quad (3.91)$$

whence  $\tau_2^{(r)}$  can be found, again up to a function of  $x$  only, through (3.84):

$$\tau_2^{(r)} = C_2^{(r)}(x) + C_{1,x}^{(r)}z - 2 \int_b^z \tau_{1,x}^{(r-1)}(x, z') dz'. \quad (3.92)$$

As before,  $C_1^{(r)}$  and  $C_2^{(r)}$  have to be found through matching with the boundary layer.  $u^{(r)}$  at the base  $z = b$  can then be found through solving (3.90), and  $u^{(r)}(z)$  can then be computed by integrating (3.87).  $v^{(r)}$  follows from (3.86) and (3.89). Note also that  $\tau_1$  only appears up to the correction term  $\tau_1^{(r-1)}$  in (3.84)–(3.90). Hence, in order to compute corrections to  $\tau_2$ ,  $p$ ,  $u$  and  $v$  up to  $\tau_2^{(N)}$ ,  $p^{(N)}$ ,  $u^{(N)}$  and  $v^{(N)}$ , it suffices that (3.82) allows us to compute  $\tau_1$  to order  $N - 1$ .

In order to construct an outer expansion for  $u$  up to an error of  $O(\varepsilon^2)$ , we still have to deal with the terms indicated by superscripts  $(i, 1)$  and  $(i, 2)$  in (3.73). These arise purely due to the boundary layer, and are not the result of higher order corrections in the outer region. From expanding (3.6), (3.9) and (3.11) (in particular, Taylor expanding the viscosity function in (3.9) and the friction law in (3.11), using the ansatz (3.73)), we find the simple relations

$$\begin{aligned} \tau_{2,z}^{(i,1)} = 0, \quad u_z^{(i,1)} = 0, \quad \left( m|u^{(0)}|^{m-1}u^{(i,1)} \right) \Big|_{z=b} = \tau_2^{(i,1)} \Big|_{z=b}, \\ u_z^{(i,2)} = 2n|\tau_2^{(0)}|^{n-1}\tau_1^{(i,1)}, \quad m|u^{(0)}|^{m-1}u^{(i,2)} = 0. \end{aligned} \quad (3.93)$$

Denoting  $u_b^{(0)} = u^{(0)}|_{z=b}$ , these have solution

$$\tau_2^{(i,1)} = C_i(x), \quad u^{(i,1)} = \frac{C_i(x)}{m|u_b^{(0)}|^{m-1}}, \quad u^{(i,2)} = 2C_i(x) \int_b^z n|\tau_2^{(0)}(x, z')|^{n-1} dz', \quad (3.94)$$

where the unknown function  $C_i$  must be determined once more through matching with the inner solution.

We consider the inner solution next, but before we do so, we wish to underline why the boundary layer treatment is necessary in the first place, based on our results for the outer problem so far. If we were to suppose that (3.73) furnishes an asymptotic expansion uniformly valid in the fluid domain, then applying the boundary conditions (3.3) and (3.4) directly would yield  $p^{(0)} = \tau_2^{(0)} = 0$  on  $z = h$ , and hence

$$C_1^{(0)} = h, \quad C_2^{(0)} = -hh_x. \quad (3.95)$$

Using this in the leading order solution (3.81) leads to

$$\begin{aligned} p^{(0)} &= h - z, & \tau_2^{(0)} &= -h_x(h - z), \\ u^{(0)} &= -(h - b)^{1/m} |h_x|^{1/m-1} h_x - 2\varepsilon \lambda^n \frac{|h_x|^{n-1} h_x}{n+1} [(h - b)^{n+1} - (h - z)^{n+1}], \\ \tau_1^{(0)} &= -\frac{[(h - b)^{1/m} |h_x|^{1/m-1} h_x]_x}{(h - z)^{n-1} |h_x|^{n-1}} \\ &\quad - 2\varepsilon \lambda^n \frac{nh_{xx} [(h - z)^{n+1} - (h - b)^{n+1}] + (n+1)h_x [(h - z)^n h_x - (h - b)^n (h_x - b_x)]}{(n+1)(h - z)^{n-1}}. \end{aligned} \quad (3.96)$$

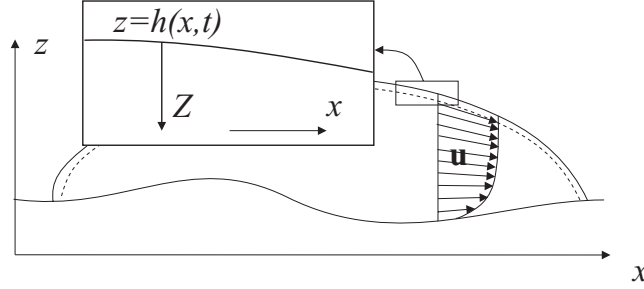
This is, of course, nothing more than the classical lubrication approximation, and as we shall see, it does give the correct solution at leading order. However, with  $n > 1$ ,  $\tau_1^{(0)}$  becomes singular at  $z = h$  as  $\tau_1^{(0)} \sim (h - z)^{1-n}$ , and this is at the heart of the need for a boundary layer treatment. This becomes amply clear when we go to higher order. For instance, with  $r = 1$ , we have  $\gamma_1$  given by (3.83), and hence

$$\tau_1^{(1)} = \frac{(n-1)u_x^{(0)3}}{2|\tau_2^{(0)}|^{3n-1}}.$$

However, from (3.96), we know that the classical lubrication solution predicts  $\tau_2^{(0)} \sim (h - z)$  near  $z = h$ , and hence  $\tau_1^{(1)} \sim (h - z)^{1-3n}$ . Hence  $\tau_1^{(1)}$  is more singular than  $\tau_1^{(0)}$ , and the presumed ordering of the asymptotic expansion (3.73) breaks down. To remedy this, the following boundary layer treatment is necessary.

### 3.5.2 The inner problem

The spectre that hangs over the determination of  $C_1^{(r)}(x)$ ,  $C_2^{(r)}(x)$  and  $C_i(x)$  above is the matching of solutions with the surface boundary layer. The boundary layer is the region in which shear stress  $\tau_2$  becomes small, leading to high viscosity. In fact, the thickness of the boundary layer is the region in which both, shear stress and normal stress contribute comparably to the viscosity term  $(\tau_1^2 + \lambda^2 \tau_2^2)^{(n-1)/2}$ . Normal stress  $\tau_1$  in the boundary layer is of  $O(1)$  (while (3.73) puts  $\tau_1 \sim \lambda^{1-n}$  in the outer region), and hence we anticipate  $\tau_2 \sim \lambda^{-1}$  in the boundary layer. The classical lubrication (3.96) for the outer solution predicts  $\tau_2 \sim -h_x(h - z)$  and hence  $\tau_2 \sim \lambda^{-1}$  within a distance of  $O(\lambda^{-1})$  of the upper surface. This is also consistent with the boundary layer thickness deduced in section 3 of



**Fig. 2** Illustration of the inner and outer domains (indicated by the dashed line) on the  $Z$ -coordinate used in the boundary layer. The boundary layer occurs where the velocity profile, indicated by arrows on the right, exhibits little vertical shear  $u_z$ .

(28) for the case  $n = 3$  and  $\lambda = \varepsilon^{-1/n}$ , bearing in mind that the aspect ratio in (28) is denoted by  $\delta$ .

These considerations lead to the rescaling

$$\begin{aligned} Z &= \lambda(h - z), & X &= x, & T_1 &= \tau_1 & T_2 &= \lambda\tau_2, \\ P &= \lambda p, & U &= u, & V &= v. \end{aligned} \quad (3.97)$$

with the corresponding transformation of derivatives

$$\frac{\partial}{\partial z} = -\lambda \frac{\partial}{\partial Z}, \quad \frac{\partial}{\partial x} = \frac{\partial}{\partial X} + \lambda \frac{\partial h}{\partial x} \frac{\partial}{\partial Z}. \quad (3.98)$$

Under this rescaling, Blatter's equations (3.6)–(3.9) and surface boundary conditions (3.4) and (3.12) become:

$$\varepsilon h_x T_{1,Z} + \frac{\varepsilon}{\lambda} T_{1,X} - T_{2,Z} - h_x P_Z - \frac{1}{\lambda} P_X = 0, \quad (3.99)$$

$$\varepsilon T_{1,Z} + P_Z - 1 = O(\varepsilon^2), \quad (3.100)$$

$$U_X + \lambda(h_x U_Z - V_Z) = 0, \quad (3.101)$$

$$U_X + \lambda h_x U_Z = (T_1^2 + T_2^2)^{(n-1)/2} T_1, \quad (3.102)$$

$$U_Z = -2 \frac{\varepsilon}{\lambda} (T_1^2 + T_2^2)^{(n-1)/2} T_2, \quad (3.103)$$

in the boundary layer domain  $0 < Z \ll \lambda$ , while at the upper surface  $Z = 0$

$$(P - \varepsilon T_1) h_x + T_2 = 0, \quad (3.104)$$

$$-P - \varepsilon T_1 = O(\varepsilon^2). \quad (3.105)$$

Above, we have merely assumed that the omitted  $O(\varepsilon^2)$  terms in Blatter's model remain of  $O(\varepsilon^2)$  under the rescaling. As indicated at the beginning of section 3.5, this still needs to be verified using the corresponding equations in the full Stokes model, namely (2.23) and

(2.30). Under the rescaling, these become

$$\varepsilon T_{1,Z} + P_Z + \varepsilon^2 \left( \frac{1}{\lambda} T_{2,X} + h_x - T_{2,Z} \right) - 1 = 0 \quad \text{on } Z > 0, \quad (3.106)$$

$$-P - \varepsilon T_1 - \frac{\varepsilon^2}{\lambda} T_2 h_x = 0 \quad \text{on } Z = 0, \quad (3.107)$$

which confirms that (3.100) and (3.105) are indeed correct to an error of  $O(\varepsilon^2)$ .

Assuming that we have scaled all dependent variables correctly, in the sense that they have  $O(1)$  leading-order contributions, we can now expand to capture these leading order terms and subsequent higher order corrections up to an error of  $O(\varepsilon^2)$ ,

$$\begin{aligned} T_1 &\sim T_1^{(0)} + O(\varepsilon), \\ T_2 &\sim T_2^{(0)} + \varepsilon T_2^{(1)} + \varepsilon \lambda^{-1} T_2^{(2)} + O(\varepsilon^2), \\ P &\sim P^{(0)} + \varepsilon P^{(1)} + O(\varepsilon^2), \\ U &\sim U^{(0)} + \varepsilon \lambda^{-1} U^{(1)} \\ &\quad + \varepsilon \lambda^{-2} U^{(o,1)} + \varepsilon^2 \lambda^{n-2} U^{(o,2)} + \varepsilon \lambda^{-n} \left( U^{(o,3)} + \lambda^{-2n} U^{(o,4)} + \dots \right) + O(\varepsilon^2), \\ V &\sim V^{(0)} + \lambda^{-1} V^{(1)} + \varepsilon \lambda^{-1} V^{(2)} + \varepsilon \lambda^{-3} V^{(o,1)} \\ &\quad + \varepsilon^2 \lambda^{n-3} V^{(o,2)} + \varepsilon \lambda^{-n-1} \left( V^{(o,3)} + \lambda^{-2n} V^{(o,4)} + \dots \right) + O(\varepsilon^2). \end{aligned} \quad (3.108)$$

where superscripts (1), (2) etc. again indicate terms that arise from expanding (3.99)–(3.103), while superscripts (o, 1), (o, 2) etc. indicate terms that arise from matching with the outer solution.

At leading order, we find

$$-T_{2,Z}^{(0)} - h_x P_Z^{(0)} = 0, \quad (3.109)$$

$$P_Z^{(0)} - 1 = 0, \quad (3.110)$$

$$h_x U_Z^{(0)} - V_Z^{(0)} = 0, \quad (3.111)$$

$$U_X^{(0)} = \left( T_1^{(0)2} + T_2^{(0)2} \right)^{(n-1)/2} T_1^{(0)}, \quad (3.112)$$

$$U_Z^{(0)} = 0 \quad (3.113)$$

in the inner domain, with boundary conditions at  $Z = 0$ :

$$T_2^{(0)} = 0, \quad (3.114)$$

$$P^{(0)} = 0. \quad (3.115)$$

This has solution

$$P^{(0)} = Z, \quad T_2^{(0)} = -h_x Z, \quad U^{(0)} = C_3^{(0)}(X), \quad (3.116)$$

while  $T_1^{(0)}$  is determined implicitly by (3.112) once  $C_3^{(0)}(X)$  has been found, in this case by matching with the outer problem.

To proceed to higher order, we need to recognize that  $P_X^{(0)} = 0$  and hence that the last term on the left-hand side of (3.99) only contributes at  $O(\varepsilon/\lambda)$ , while  $U_Z^{(0)} = 0$  and hence the second term on the left-hand side of (3.102) only appears at  $O(\varepsilon)$ . The first order correction therefore satisfies

$$h_x T_{1,Z}^{(0)} - T_{2,Z}^{(1)} - h_x P_Z^{(1)} = 0, \quad (3.117)$$

$$T_{1,Z}^{(0)} + P_Z^{(1)} = 0, \quad (3.118)$$

$$U_X^{(0)} = V_Z^{(1)}, \quad (3.119)$$

$$U_Z^{(1)} = -2 \left( T_1^{(0)2} + T_2^{(0)2} \right)^{(n-1)/2} T_2^{(0)}, \quad (3.120)$$

in  $Z > 0$ , with boundary conditions at  $Z = 0$  that read

$$P^{(1)} h_x + T_2^{(1)} - T_1^{(0)} h_x = 0, \quad (3.121)$$

$$-P^{(1)} - T_1^{(0)} = 0. \quad (3.122)$$

Having computed  $T_1^{(0)}$  previously, we find  $P^{(1)} = -T_1^{(0)}$  from (3.118) and (3.122), and  $T_2^{(1)}$  can then be computed from (3.117) and (3.121). Specifically, we find

$$T_2^{(1)} = 2h_x T_1^{(0)}. \quad (3.123)$$

Lastly,  $U^{(1)}$  can be found up to a function of  $x$  from (3.120); this function must again be found by matching with the outer problem. Specifically, we have

$$U^{(1)}(X, Z) = C_3^{(1)}(X) - 2 \int_0^Z \left( T_1^{(0)}(X, Z')^2 + T_2^{(0)}(X, Z')^2 \right)^{(n-1)/2} T_2^{(0)}(X, Z') dZ'. \quad (3.124)$$

Meanwhile, the second order correction for  $T_2$  satisfies

$$T_{1,X}^{(0)} - T_{2,Z}^{(2)} - P_X^{(1)} = 0 \quad (3.125)$$

in  $0 < Z \ll \lambda^{-1}$ , with boundary condition  $T_2^{(2)} = 0$  on  $Z = 0$ . Having computed the terms  $T_1^{(0)}$  and  $P^{(1)}$  previously,  $T_2^{(2)}$  can then be calculated by simple quadrature:

$$T_2^{(2)}(X, Z) = \int_0^Z 2T_{1,X}^{(0)}(X, Z') dZ'. \quad (3.126)$$

Lastly, (3.103) implies that all the velocity terms  $U^{(o,1)}$ ,  $U^{(o,2)}$  etc. satisfy  $U_Z^{(o,r)} = 0$  for  $r = 1, 2, \dots$ , and hence that  $U^{(o,r)}(X, Z) = C_o^{(r)}(X)$ , where the  $C_o$ 's are functions that must be determined by matching with the outer solution.

### 3.5.3 Matching

Next, we consider the matching procedure. To save space, we spell this out explicitly only for some of the low order terms in order to demonstrate that matching is indeed feasible,

and can determine the  $C_1$ ,  $C_2$ ,  $C_3$  and  $C_i$ ,  $C_o$  functions that appear in the solutions above. To match the leading-order outer and inner solutions in (3.81) and (3.116) clearly requires

$$C_1^{(0)} = h, \quad C_2^{(0)} = -hh_x, \quad C_3^{(0)} = \lim_{z \rightarrow h} u^{(0)}. \quad (3.127)$$

In particular, with  $C_1^{(0)}$  and  $C_2^{(0)}$  as above we recover the classical lubrication approximation solution (3.96) as the leading order outer solution.

Using this, the procedure in section 3.5.1 then allows the first two terms in the outer solution (or more specifically, all corrections with superscripts (0), (1) and  $(i, 1)$ ,  $(i, 2)$  in (3.73)) to be computed as follows:

$$\tau_1 \sim \lambda^{1-n} \frac{u_x^{(0)}}{|h_x(h-z)|^{n-1}} - \lambda^{1-3n} \frac{(n-1)u_x^{(0)3}}{2|h_x(h-z)|^{3n-1}} + \dots, \quad (3.128)$$

$$\begin{aligned} \tau_2 \sim & -h_x(h-z) + \frac{\varepsilon}{\lambda^2} C_i \\ & + \frac{\varepsilon}{\lambda^n} \left\{ C_2^{(1)} + C_{1,x}^{(1)} z - 2 \int_b^z \left( \frac{u_x^{(0)}}{|h_x|^{n-1}} \right)_x \frac{1}{(h-z')^{n-1}} dz' \right. \\ & \left. + 2 \int_b^z \frac{(n-1)u_x^{(0)}}{|h_x|^{n-3} h_x} \frac{1}{(h-z')^n} dz' \right\} + \dots, \end{aligned} \quad (3.129)$$

$$p \sim h - z + \frac{\varepsilon}{\lambda^n} \left[ C_1^{(1)} - \frac{u_x^{(0)}}{|h_x(h-z)|^{n-1}} \right] + \dots, \quad (3.130)$$

$$\begin{aligned} u \sim & u^{(0)} + \frac{\varepsilon}{\lambda^2} \frac{C_i(x)}{m|u_b^{(0)}|^{m-1}} + 2\varepsilon^2 \lambda^{n-2} |h_x|^{n-1} C_i(x) [(h-b)^n - (h-z)^n] \\ & + \frac{\varepsilon}{\lambda^n} \left\{ - \int_b^z \frac{(n-1)u_x^{(0)2}}{|h_x|^{n-1} h_x (h-z')^n} dz' + \frac{C_2^{(1)} + C_{1,x}^{(1)} b}{m|u_b^{(0)}|^{m-1}} \right\} + \dots \end{aligned} \quad (3.131)$$

where  $u^{(0)}$  is given by (3.96),

$$u^{(0)} = -(h-b)^{\frac{1}{m}} |h_x|^{\frac{1}{m}-1} h_x - 2\varepsilon \lambda^n \frac{|h_x|^{n-1} h_x}{n+1} [(h-b)^{n+1} - (h-z)^{n+1}], \quad (3.132)$$

and  $u_b^{(0)} = u^{(0)}|_{z=b} = -(h-b)^{\frac{1}{m}} |h_x|^{\frac{1}{m}-1} h_x$  as before.

In order to match with the inner solution, the limiting behaviour of the outer solutions as  $z \rightarrow h$  from below must be established. Let  $Z = \lambda(h-z)$  as in the previous subsection, and define

$$u_h^{(0)} = \lim_{z \rightarrow h} u^{(0)}. \quad (3.133)$$

We can recognize from (3.127) that  $C_3^{(0)} = u_h^{(0)} = U^{(0)}$ . We have

$$\begin{aligned} u^{(0)} &= u_h^{(0)} - 2\varepsilon\lambda^n \frac{|h_x|^{n-1}h_x [(h-z)^{n+1} - (h-b)^{n+1}]}{n+1} \\ &= u_h^{(0)} + 2\varepsilon\lambda^n \frac{|h_x|^{n-1}h_x(h-b)^{n+1}}{n+1} - 2\frac{\varepsilon}{\lambda} \frac{|h_x|^{n-1}h_x Z^{n+1}}{n+1}, \end{aligned} \quad (3.134)$$

$$\begin{aligned} u_x^{(0)} &= u_{h,x}^{(0)} - 2\varepsilon\lambda^n \frac{(|h_x|^{n-1}h_x)_x [(h-z)^{n+1} - (h-b)^{n+1}]}{n+1} \\ &\quad - 2\varepsilon\lambda^n |h_x|^{n-1}h_x [(h-z)^n h_x - (h-b)^n (h_x - b_x)] \\ &= u_{h,x}^{(0)} + 2\varepsilon\lambda^n \frac{(|h_x|^{n-1}h_x)_x (h-b)^{n+1} + (n+1)|h_x|^{n-1}h_x (h-b)^n (h_x - b_x)}{n+1} \\ &\quad - 2\varepsilon|h_x|^{n+1}Z^n - 2\frac{\varepsilon}{\lambda} \frac{(|h_x|^{n-1}h_x)_x Z^{n+1}}{n+1}. \end{aligned} \quad (3.135)$$

Immediately, we see that we can replace  $u_x^{(0)}$  with  $u_{h,x}^{(0)}$  in (3.128), (3.129) and (3.130), as the  $O(\varepsilon\lambda^n)$  difference between  $u_x^{(0)}$  and  $u_{h,x}^{(0)}$  leads to a  $O(\varepsilon\lambda)$  change in  $\tau_1$  and a  $O(\varepsilon^2)$  change in  $\tau_2$  and  $p$ , and these are of the order of the omitted higher order terms in the expansion in (3.73). With this substitution, we can simplify some of the integrals in (3.129) and (3.130) as  $u_{h,x}^{(0)}$  is independent of  $Z$ . Then (3.128)–(3.131) become in terms of  $Z$

$$\tau_1 \sim \frac{u_{h,x}^{(0)}}{|h_x|^{n-1}Z^{n-1}} - \frac{(n-1)u_{h,x}^{(0)3}}{2|h_x|^{3n-1}Z^{3n-1}} + \dots, \quad (3.136)$$

$$\begin{aligned} \tau_2 &\sim -\frac{1}{\lambda}h_x Z + \frac{\varepsilon}{\lambda} \frac{2u_{h,x}^{(0)}}{|h_x|^{n-3}h_x Z^{n-1}} + \frac{\varepsilon}{\lambda^2} \left\{ C_i - \frac{2}{n-2} \left( \frac{u_{h,x}^{(0)}}{|h_x|^{n-1}} \right)_x \frac{1}{Z^{n-2}} \right\} \\ &\quad + \frac{\varepsilon}{\lambda^n} \left\{ C_2^{(1)} + C_{1,x}^{(1)}h + \left( \frac{u_{h,x}^{(0)}}{|h_x|^{n-1}} \right)_x \frac{2}{(n-2)(h-b)^{n-2}} \right. \\ &\quad \left. + \frac{2u_{h,x}^{(0)}}{(n-1)|h_x|^{n-3}h_x(h-b)^{n-1}} \right\} + \dots, \end{aligned} \quad (3.137)$$

$$p \sim \frac{1}{\lambda}Z - \frac{\varepsilon}{\lambda} \frac{u_{h,x}^{(0)}}{|h_x|^{n-1}Z^{n-1}} + \frac{\varepsilon}{\lambda^n} C_1^{(1)} + \dots, \quad (3.138)$$

$$\begin{aligned} u &\sim u_h^{(0)} + \frac{\varepsilon}{\lambda} \left( 2\frac{|h_x|^{n-1}h_x Z^{n+1}}{n+1} - \frac{u_{h,x}^{(0)2}}{|h_x|^{n-1}h_x Z^{n-1}} \right) + \frac{\varepsilon}{\lambda^2} \frac{C_i}{m|u_b^{(0)}|^{m-1}} \\ &\quad + \varepsilon^2 \lambda^{n-2} |h_x|^{n-1} (h-b)^n C_i + \frac{\varepsilon}{\lambda^n} \left\{ \frac{u_{h,x}^{(0)2}}{|h_x|^n (h-b)^{n-1}} + \frac{C_2^{(1)} + C_{1,x}^{(1)}b}{m|u_b^{(0)}|^{m-1}} \right\} + \dots \end{aligned} \quad (3.139)$$

Next, we consider the inner solution. This takes the form

$$T_1 \sim T_1^{(0)} + O(\varepsilon), \quad (3.140)$$

$$T_2 \sim -h_x Z + \varepsilon 2h_x T_1^{(0)} + \frac{\varepsilon}{\lambda} 2 \int_0^Z T_{1,X}^{(0)}(X, Z') dZ' + O(\varepsilon^2), \quad (3.141)$$

$$P \sim Z - \varepsilon T_1^{(0)} + O(\varepsilon^2), \quad (3.142)$$

$$U \sim u_h^{(0)} + \frac{\varepsilon}{\lambda} \left[ C_3^{(1)}(x) + 2 \int_0^Z \left( T_1^{(0)}(X, Z')^2 + h_x^2 Z'^2 \right)^{(n-1)/2} h_x Z' dZ' \right] \\ + \frac{\varepsilon}{\lambda^2} C_o^{(1)}(x) + \varepsilon^2 \lambda^{n-2} C_o^{(2)}(x) + \frac{\varepsilon}{\lambda^n} C_o^{(3)}(x) + O(\varepsilon \lambda^{-3n}) + O(\varepsilon^2), \quad (3.143)$$

where  $T_1^{(0)}$  is defined implicitly by

$$u_{h,x}^{(0)} = \left( T_1^{(0)2} + h_x^2 Z^2 \right)^{(n-1)/2} T_1^{(0)}. \quad (3.144)$$

For large  $Z$ , we obtain from (3.144) that  $T_1^{(0)} \sim u_{h,x}^{(0)} / (|h_x Z|^{n-1})$  at leading order (by recognizing that  $u_{h,x}^{(0)}$  is independent of  $Z$  and hence that the term  $h_x^2 Z^2$  dominates the surd). Expanding to higher order, we then find

$$T_1^{(0)} \sim \frac{u_{h,x}^{(0)}}{|h_x Z|^{n-1}} - \frac{(n-1)u_{h,x}^{(0)3}}{2|h_x Z|^{3n-1}} + \frac{(n-1)(15-n)u_{h,x}^{(0)5}}{8|h_x Z|^{5n-1}} + \dots, \quad (3.145)$$

which matches the expression for  $\tau_1$  in (3.136). Next, we consider the behaviour for large  $Z$  of  $T_2$ ,  $P$  and  $U$ . We use  $\tau_2 = T_2/\lambda$ ,  $p = P/\lambda$  and  $u = U$  and for definiteness assume that  $n > 2$  (the case  $n < 2$  can be dealt with analogously). Expanding to the two leading terms

in  $Z$ , we can write

$$\begin{aligned} \int_0^Z T_{1,X}^{(0)}(X, Z') dZ' &\sim \int_0^\infty T_{1,X}^{(0)} dZ' - \int_Z^\infty \left( \frac{u_{h,x}^{(0)}}{|h_x Z'|^{n-1}} \right)_X dZ' \\ &= \int_0^\infty T_{1,X}^{(0)} dZ' - \left( \frac{u_{h,x}^{(0)}}{|h_x|^{n-1}} \right)_x \frac{2}{(n-2)Z^{n-2}}, \end{aligned} \quad (3.146)$$

$$\begin{aligned} \int_0^Z \left( T_1^{(0)2} + h_x^2 Z'^2 \right)^{(n-1)/2} h_x Z' dZ' &= \int_0^Z \left( T_1^{(0)2} + h_x^2 Z'^2 \right)^{(n-1)/2} h_x Z' - |h_x|^{n-1} h_x Z'^n dZ' \\ &\quad + \frac{|h_x|^{n-1} h_x Z^{n+1}}{n+1} \\ &\sim \int_0^\infty \left( T_1^{(0)2} + h_x^2 Z'^2 \right)^{(n-1)/2} h_x Z' - |h_x|^{n-1} h_x Z'^n dZ' \\ &\quad + \frac{|h_x|^{n-1} h_x Z^{n+1}}{n+1} - \int_Z^\infty \frac{(n-1)T_1^{(0)2} |h_x|^{n-3} h_x Z'}{2} dZ' \\ &\sim \int_0^\infty \left( T_1^{(0)2} + h_x^2 Z'^2 \right)^{(n-1)/2} h_x Z' - |h_x|^{n-1} h_x Z'^n dZ' \\ &\quad + \frac{|h_x|^{n-1} h_x Z^{n+1}}{n+1} - \frac{u_{h,x}^{(0)2}}{2|h_x|^{n-1} h_x Z^{n-1}}, \end{aligned} \quad (3.147)$$

where it is clear from (3.145) that the integrals taken over  $(0, \infty)$  converge because we have assumed that  $n > 2$ . Hence we have the following limiting behaviour for the inner solution:

$$\begin{aligned} \tau_2 &\sim -\frac{1}{\lambda} h_x Z + \frac{\varepsilon}{\lambda} \frac{2u_{h,x}^{(0)}}{|h_x|^{n-3} h_x Z^{n-1}} \\ &\quad + \frac{\varepsilon}{\lambda^2} \left\{ 2 \int_0^\infty T_{1,X}^{(0)} dZ - \frac{2}{n-2} \left( \frac{u_{h,x}^{(0)}}{|h_x|^{n-1}} \right)_x \frac{1}{Z^{n-2}} \right\} + \dots, \end{aligned} \quad (3.148)$$

$$p \sim \frac{Z}{\lambda} - \frac{\varepsilon}{\lambda} \frac{u_{h,x}^{(0)}}{|h_x Z|^{n-1}} + \dots, \quad (3.149)$$

$$\begin{aligned} u &\sim u_h^{(0)} + \frac{\varepsilon}{\lambda} \left\{ C_3^{(1)}(x) + \frac{2|h_x|^{n-1} h_x Z^{n+1}}{n+1} - \frac{u_{h,x}^{(0)2}}{|h_x|^{n-1} h_x Z^{n-1}} \right. \\ &\quad \left. + 2 \int_0^\infty \left[ \left( T_1^{(0)2} + h_x^2 Z'^2 \right)^{(n-1)/2} h_x Z' - |h_x|^{n-1} h_x Z'^n \right] dZ' \right\} \\ &\quad + \frac{\varepsilon}{\lambda^2} C_o^{(1)}(x) + \varepsilon^2 \lambda^{n-2} C_o^{(2)}(x) + \frac{\varepsilon}{\lambda^n} C_o^{(3)}(x) + \dots \end{aligned} \quad (3.150)$$

Comparing (3.137)–(3.139) with (3.148)–(3.150), the solutions match as required if

$$C_i = 2 \int_0^\infty T_{1,X}^{(0)} dZ, \quad C_o^{(1)} = \frac{C_i}{m|u_b^{(0)}|^{m-1}}, \quad C_o^{(2)} = |h_x|^{n-1} C_i (h-b)^n,$$

$$\begin{aligned}
C_o^{(3)} &= \frac{u_x^{(0)2}}{|h_x|^n(h-b)^{n-1}} + \frac{C_2^{(1)}}{m|u_b^{(0)}|^{m-1}}, & C_1^{(1)} &= 0, \\
C_2^{(1)} &= - \left( \frac{u_{h,x}^{(0)}}{|h_x|^{n-1}} \right)_x \frac{2}{(n-2)(h-b)^{n-2}} + \frac{2u_{h,x}^{(0)}}{|h_x|^{n-3}h_x(n-1)(h-b)^{n-1}} \\
C_3^{(1)} &= -2 \int_0^\infty \left[ \left( T_1^{(0)2} + h_x^2 Z'^2 \right)^{(n-1)/2} h_x Z' - |h_x Z'|^{n-1} h_x Z'^n \right] dZ', & (3.151)
\end{aligned}$$

and the higher-order functions  $C_1^{(r)}$ ,  $C_2^{(r)}$  and  $C_o^{(r)}$  can be obtained by expanding further. We see in particular that the ‘matching terms’  $\tau_1^{(i,1)}$ ,  $u^{(i,1)}$  and  $u^{(i,2)}$  in the outer expansion (3.73) are fully determined through this matching exercise, as they are given in terms of  $C_i = 2 \int_0^\infty T_{1,X}^{(0)} dZ$  through (3.94).

We have restricted ourselves to only the first few terms in the inner and outer expansions above, and hope the reader will appreciate that proceeding to a general order  $r$  is possible in principle but neither feasible nor desirable in this paper. That is to say, there is no reason to believe that higher order terms will cause problems, but equally, there appears to be no straightforward way of expanding to a general order  $r$ . As stated before, our aim in constructing the expansions above was to demonstrate that it is possible to use Blatter’s equations to find an asymptotic expansion for the velocity  $u$  in both, the inner and outer domain, to an error of  $O(\varepsilon^2)$ , and that only corrections of  $O(\varepsilon^2)$  and higher would require the retention of terms in the Stokes equations that are omitted in Blatter’s model.

The demonstration that Blatter’s model suffices to determine  $u$  to  $O(\varepsilon^2)$  was our main objective in the analysis above. Our results go further, however, as they provide a more complete analysis of the high-viscosity boundary layer that forms near the surface of a shear-thinning power-law fluid flowing as a thin film than has previously been given by Johnson & McMeeking (28). The most important insight we have gained is that the high-viscosity boundary layer introduces a correction to shear stress  $\tau_2$  in the outer solution of  $O(\varepsilon\lambda^{-2})$ , namely the term  $\tau_2^{(i,1)}$  in (3.73)<sub>2</sub>. (This term was, in fact, not included in the outer expansion in Johnson & McMeeking’s paper, their equation (64), as they do not carry out their matching procedure to this order, although their leading order inner solution, equations (72)–(73) of Johnson & McMeeking’s paper, does correspond to the leading order inner problem (3.109)–(3.115) here). The shear stress correction  $\tau_2^{(i,1)}$  is essentially the result of elevated normal stresses acting parallel to the upper surface, as is evident from the result  $\tau_2^{(i,1)} = C_i(x) = 2 \int_0^\infty T_{1,X}^{(0)} dZ$  deduced for  $n > 2$  in (3.151) above. The magnitude of the surface normal stress  $T_1^{(0)}$  itself is controlled by surface velocities  $u_h^{(0)}$  through (3.144). In turn, the  $O(\varepsilon\lambda^{-2})$  shear stress correction  $\tau_2^{(i,1)}$  introduces a correction of  $O(\varepsilon\lambda^{-2})$  into the velocity field, represented by  $u^{(i,1)}$  and  $u^{(i,2)}$  in (3.73).

As already pointed out in (28), it is important to realize that the surface boundary layer does not affect the flow at leading order, for which the standard lubrication approximation with solution (3.96) is appropriate. Higher order corrections to the leading order solution  $u^{(0)}$  can come not only from matching with the boundary layer, which contributes the terms  $u^{(i,1)}$  and  $u^{(i,2)}$  in (3.73), but also from higher order terms within the outer region, which contribute the terms  $u^{(1)}$ ,  $u^{(2)}$  etc. For  $n > 2$  (the case most applicable in glaciology, as ice is usually treated as a power-law fluid with  $n \approx 3$ ), the  $O(\varepsilon\lambda^{-2})$  velocity correction due to

the boundary layer is in fact the dominant correction to  $u^{(0)}$ , while for  $n < 2$ , the dominant correction is the  $O(\varepsilon\lambda^{-n})$  term  $u^{(1)}$  due to higher order stresses in the outer region.

In closing this section, we again note that the normal stress term  $\tau_1$  does not need to be computed to  $O(\varepsilon^2)$  in order to find horizontal velocity to  $O(\varepsilon^2)$  for the parameter regime considered here: in the outer region, we require  $\tau_1$  to an error of  $O(\varepsilon\lambda)$ , while in the boundary layer, we were able to make do with calculating  $\tau_1 = T_1$  only to an error of  $O(\varepsilon)$ .

### 3.6 Slow sliding (ii): the case $\lambda \gg \varepsilon^{-1/n}$

This last parameter regime to be considered corresponds to the situation in which the contribution of shearing to fluid velocity  $u$  dominates over sliding. A rescaling once more becomes necessary when this is the case, i.e., when  $\lambda \gg \varepsilon^{-1/n}$ . This is apparent because the rate of shear predicted by (3.9) is large in this parameter regime for an  $O(1)$  shear stress  $\tau_1$ . The reason for this is simple: in the scale relation (2.14), we have chosen a velocity scale  $[u]$  based on the friction coefficient  $C$  of the bed, and this is no longer appropriate here; the natural velocity scale  $[u]$  for slow sliding should be the shear velocity  $[u_s]$  given in equation (2.19).

Re-introducing asterisks on the scaled variables, the necessary rescaling takes the form

$$\begin{aligned} \tau_1^{***} &= \varepsilon^{-1/n} \lambda^{-1} \tau_1^*, & \tau_2^{***} &= \tau_2^*, & p^{***} &= p^*, \\ u^{***} &= \varepsilon^{-1} \lambda^{-n} u^*, & v^{***} &= \varepsilon^{-1} \lambda^{-n} v^*, & a^{***} &= \varepsilon^{-1} \lambda^{-n} a^*, & x^{**} &= x^*, \\ z^{***} &= z^*, & h^{***} &= h^*, & b^{***} &= b^*, \end{aligned} \quad (3.152)$$

and we define a sliding parameter  $\gamma$  through

$$\gamma = \varepsilon^{-m} \lambda^{-mn}. \quad (3.153)$$

Given that the parameter range considered here is  $\lambda \gg \varepsilon^{-1/n}$ , it is clear that  $\gamma \ll 1$ . Substituting the rescaled variables into (2.22)–(2.31) and immediately dropping the asterisks again, we obtain the following: in  $b < z < h$

$$\frac{\varepsilon}{\varepsilon^{-1/n}} \tau_{1,x} + \tau_{2,z} - p_x = 0, \quad (3.154)$$

$$-\frac{\varepsilon}{\varepsilon^{-1/n}} \tau_{1,z} + \varepsilon^2 \tau_{2,x} - p_z - 1 = 0, \quad (3.155)$$

$$u_x + v_z = 0, \quad (3.156)$$

$$u_x = \left( \tau_1^2 + \varepsilon^{-2/n} \tau_2^2 \right)^{(n-1)/2} \tau_1, \quad (3.157)$$

$$u_z + \varepsilon^2 v_x = 2\varepsilon^{1-1/n} \left( \tau_1^2 + \varepsilon^{-2/n} \tau_2^2 \right)^{(n-1)/2} \tau_2, \quad (3.158)$$

while boundary conditions at the base  $z = b$  are

$$v = ub_x, \quad (3.159)$$

$$\gamma \frac{-\frac{\varepsilon}{\varepsilon^{-1/n}} b_x \tau_1 + (1 - \varepsilon^2 b_x^2) \tau_2}{\sqrt{1 + \varepsilon^2 b_x^2}} = (1 + \varepsilon^2 b_x^2)^{\frac{m}{2}} |u|^{m-1} u, \quad (3.160)$$

and at the upper surface  $z = h$  we have

$$\left(p - \frac{\varepsilon}{\varepsilon^{-1/n}} \tau_1\right) h_x + \tau_2 = 0, \quad (3.161)$$

$$-p - \frac{\varepsilon}{\varepsilon^{-1/n}} \tau_1 - \varepsilon^2 \tau_2 h_x = 0, \quad (3.162)$$

$$h_t + u h_x = v + a. \quad (3.163)$$

This rescaled model can be recognized simply as equations (2.22)–(2.31) with the stress parameter  $\lambda$  set to  $\varepsilon^{-1/n}$ , and a small parameter  $\gamma$  appended to the left-hand side of (2.28). This small parameter indicates that, for  $O(1)$  shear stresses at the base of the ice, sliding velocities are slow. From this starting point, Blatter’s model can be obtained once more by omitting the same terms of  $O(\varepsilon^2)$  as those which led to (3.6)–(3.13), and we do not repeat the procedure here.

To show in detail that Blatter’s model, rescaled as indicated above, reproduces the same answer to an error of  $O(\varepsilon^2)$  as the Stokes equations (3.154)–(3.162) would be a formidable task but for the fact that the relevant asymptotic expansions are in essence the same as in section 3.5. The rescaling (3.152) produces Blatter’s model in the form of equations (3.1)–(3.4) and (3.7)–(3.11) with  $\lambda = \varepsilon^{-1/n}$  (a case covered in section 3.5) and a small coefficient  $\gamma$  multiplying the left-hand side of (3.11). Again, a boundary layer treatment is necessary to account for high viscosity near the upper surface, but to make our task simpler, the outer solution still can be expanded in an asymptotic expansion of the form (3.73) with  $\lambda = \varepsilon^{-1/n}$  (which allows us to omit all the higher-order terms with superscripts (1), (2) etc., as these are now all of  $O(\varepsilon^2)$ ). In fact, in expanding in this way, corrections of  $O(\gamma)$  are retained at leading order, but this does not affect the form of the solution to  $O(\varepsilon^2)$ . Likewise, the inner solution can still be expanded in the form (3.108), again with  $\lambda = \varepsilon^{-1/n}$ . In both cases, the solution procedure for inner and outer solutions and the matching of the two solutions follows exactly the same steps as in section 3.5, and can be used to show that Blatter’s model continues to produce the same solution to  $O(\varepsilon^2)$  as the original Stokes equations. Naturally, given the laborious nature of the analysis in section 3.5, we do not reproduce these steps here, and trust that interested readers will be able to fill in the details — which complete our asymptotic analysis of Blatter’s model — for themselves.

#### 4. Discussion and outlook: towards a depth-integrated higher order model

The previous section has demonstrated that Blatter’s model is a valid thin-film approximation to the Stokes equations that preserves velocity structure to an error of  $O(\varepsilon^2)$ , regardless of the slip parameter  $\lambda$ . Not only that, but we have also seen that all the usual classical thin film approximations — the lubrication approximation and viscous membrane models — can be derived from Blatter’s model itself, so Blatter’s model can be thought of as hybrid between membrane and lubrication models that retains a uniform validity for all sliding regimes, rather than describing only slow or only fast sliding. Moreover, Blatter’s model also captures the high-viscosity boundary layer at the upper surface of the shear-thinning thin film, and even in the limit of slow sliding represents an improvement over the lubrication approximation, which does not include this boundary layer.

It is important to underline that Blatter’s model remains a thin film model, and requires that the fluid film in question should have a low aspect ratio, with significant changes in stress and velocity happening over horizontal length scales much larger than film thickness.

While this is generally true in glacier and ice sheet dynamics, we emphasize that Blatter’s model can fail in narrow transition zones (boundary layers) between regions with different sliding behaviour. If a transition from, say, no slip to free slip at the base occurs over a horizontal length scale comparable with film thickness, then normal stress in the vertical direction can no longer be expected to be hydrostatic as required by Blatter’s model (see equation (3.5) in the present paper). This can have significant consequences for the coupling between these different regions, as has been demonstrated for the transition from grounded to floating ice (39). Higher order models such as Blatter’s therefore cannot necessarily be applied if such transition zones are present, and the full Stokes equations may need to be solved. Similarly, the full Stokes equations may have to be solved close to any contact lines to resolve the velocity field there. Owing to the power-law rheology that is also responsible for the complicated boundary layer structure in section 3.5, the thin film approach pursued here also fails in the slow sliding parameter regime of 3.5 when the surface slope  $h_x$  vanishes and when viscosity therefore becomes very large throughout the fluid layer; regions of this type are known in glaciology as *ice divides* and have been considered previously by various authors (30, 40, 41).

However, even as a thin-film model, Blatter’s model has some drawbacks not shared by more traditional approaches to thin film flow. The most obvious is perhaps that, in order to furnish a single thin-film model capable of describing slow and fast sliding, we were forced in some parameter regimes to retain higher order terms selectively while dropping other terms of the same order (see for instance equation (3.67)). More significant, however, is the fact that Blatter’s model cannot be depth-integrated. Although cheaper to solve numerically than the underlying Stokes equations (because a single variable  $u$  rather than the full velocity field  $(u, v)$  as well as the pressure variable  $p$  must be solved for), Blatter’s model is much less efficient as a numerical tool than either the classical lubrication approximation or viscous membrane models as the full fluid domain must be resolved.

#### 4.1 A depth-integrated model of nearly equal accuracy to Blatter’s model

A great deal of recent work has been invested in developing higher order models that perform as well, or at least nearly as well, as Blatter’s model but that can be depth-integrated (24). We consider one such model below, which probably incurs the smallest error (in an asymptotic sense) of all the higher-order models that can be depth-integrated. A complete analysis of the model is beyond the scope of the present paper. However, one of the by-products of the work in section 3 is that we have the complete asymptotic structure of the solution to Stokes’ equations up to  $O(\varepsilon^2)$  available to us, where  $\varepsilon$  is the aspect ratio of the fluid film. We will use this knowledge to sketch the basic steps of a more complete analysis of the depth-integrated model we are about to describe, and state the main results of this analysis.

The starting point for our depth-integrated model is the realization that normal stress  $\tau_1$  typically need not be determined to  $O(\varepsilon^2)$  in order to compute  $u$  to that order, as noted at the end of each subsection in section 3 (see also the ‘L1L2’ model in (26)). The constitutive relation that determines  $\tau_1$  in Blatter’s model in terms of the velocity field is (3.8),

$$u_x = (\tau_1^2 + \lambda^2 \tau_2^2)^{(n-1)/2} \tau_1.$$

We re-write this in a form that lends itself to the construction of a depth-integrated model

along the lines of the ‘L1L2’ model in (24). Let

$$u_b = u|_{z=b}$$

be velocity at the base of the fluid film, and

$$\tilde{\tau}_2 = -h_x(h - z)$$

be the shear stress predicted by the lubrication approximation. (3.8) can then be put in the form

$$u_{b,x} = (\tau_1^2 + \lambda^2 \tilde{\tau}_2^2)^{(n-1)/2} \tau_1 + r, \quad (4.1)$$

where the remainder  $r$  is given by

$$r = [u_{b,x} - u_x] - \left[ (\tau_1^2 + \lambda^2 \tilde{\tau}_2^2)^{(n-1)/2} \tau_1 - (\tau_1^2 + \lambda^2 \tau_2^2)^{(n-1)/2} \tau_1 \right]. \quad (4.2)$$

Anticipating that the remainder is always small, we omit  $r$  from (4.1). A depth-integrated model for basal velocity  $u_b$  can then be constructed from (4.1) and the rest of Blatter’s model, consisting of equations (3.6), (3.9), (3.11) and (3.12). Specifically, (4.1) *without* the remainder term defines normal stress implicitly as a function of  $u_{b,x}$  and the known shear stress  $\tilde{\tau}_2(x, z)$ :

$$\tau_1 = 2\mu(|u_{b,x}|, |\tilde{\tau}_2|)u_{b,x}. \quad (4.3)$$

An analytical formula for the effective viscosity function  $\mu$  cannot generally be given (except for  $n = 1, 2$ ), but several properties are simple to demonstrate:  $\mu(|u_{b,x}|, |\tilde{\tau}_2|)u_{b,x}$  is an increasing function of  $u_{b,x}$  and a decreasing function of  $|\tilde{\tau}_2|$ , with  $\mu$  behaving as  $\mu(|u_{b,x}|, |\tilde{\tau}_2|) \sim 1/(2|\tilde{\tau}_2|^{(n-1)/2})$  when  $\lambda|u_{b,x}|^{1/n}/|\tilde{\tau}_2| \ll 1$ , and as  $\mu(|u_{b,x}|, |\tilde{\tau}_2|) \sim |u_{b,x}|^{1/n-1}/2$  when  $\lambda|u_{b,x}|^{1/n}/|\tilde{\tau}_2| \gg 1$ .

The advantage of (4.3) is that normal stress  $\tau_1$  is now approximated as a function of the velocity gradient  $u_{b,x}$  at the base of the fluid film only, and the velocity structure in the interior of the domain need not be known in order to solve for  $u_b$ . Substituting (4.3) in the force balance equation (3.6) and integrating from  $z = b(x)$  to  $z = h(x)$  gives a relation analogous to MacAyeal’s model (3.39) on application of the boundary conditions (3.11) and (3.12):

$$4\frac{\varepsilon}{\lambda} \left[ \left( \int_b^h \mu(|u_{b,x}|, |\tilde{\tau}_2|) dz \right) u_{b,x} \right]_x - |u_b|^{m-1} u_b - (h - b)h_x = 0. \quad (4.4)$$

As  $\tilde{\tau}_2 = -h_x(h - z)$ , the term in square brackets on the left can be treated as a function of basal velocity gradient  $u_{b,x}$ , film thickness  $h - b$  and surface slope  $h_x$  alone, and moreover this function increases with  $u_{b,x}$ . Hence (4.4) can be recognized as an elliptic equation for the basal velocity  $u_b$ . This equation is posed on a one-dimensional domain, and although the viscosity term  $\int_b^h \mu(|u_{b,x}|, |\tilde{\tau}_2|) dz$  cannot be computed analytically, (4.4) is computationally much cheaper to solve than Blatter’s model (3.6), (3.8), (3.9), (3.11) and (3.12), which is posed on a two-dimensional domain.

Once the basal velocity  $u_b(x)$  has been solved for, the velocity field  $u(x, z)$  in the interior of the domain can be found *a posteriori* by integrating (3.9). To do so, we first need to calculate  $\tau_2$  from (3.6) and (3.12). With  $\tau_1$  given by (4.3), we have

$$\tau_2(x, z) = -h_x(h - z) + 4\frac{\varepsilon}{\lambda} \left[ \left( \int_z^h \mu(|u_{b,x}(x)|, |\tilde{\tau}_2(x, z')|) dz' \right) u_{b,x} \right]_x. \quad (4.5)$$

$u(x, z)$  follows from (3.9) as

$$u(x, z) = u_b(x) + \varepsilon \lambda \int_b^z \left\{ [2\mu(|u_{b,x}(x)|, |\tilde{\tau}_2(x, z')|) u_{b,x}]^2 + \lambda^2 \tau_2(x, z')^2 \right\}^{(n-1)/2} \tau_2(x, z') dz'. \quad (4.6)$$

Much has been said in the geophysical, fluid dynamics and the numerical analysis literature about the variational structure of Blatter's model and of the membrane models that describe rapid ice flow (**15**, **33**, **36**, **37**, **42**, **43**). The variational structure of these models greatly facilitates their numerical solution, as it allows minimization algorithms to be applied. It is therefore worth underlining that the depth-integrated model (4.4) also preserves this variational structure: in fact, (4.4) is the Euler-Lagrange equation for the functional

$$J(u) = \int_{\Omega} \left[ G(|u_{b,x}|; h-b, h_x) + \frac{C}{m+1} |u_b|^{m+1} - (h-b)h_x u_b \right] dx \quad (4.7)$$

where  $\Omega$  is the projection of the fluid flow domain onto the  $x$ -axis, and

$$G(|u_{b,x}|; h-b, h_x) = 4 \frac{\varepsilon}{\lambda} \int_0^{|u_{b,x}|} \left[ \int_0^{h-b} \mu(s, |(h-b-z')h_x|) dz' \right] s ds; \quad (4.8)$$

a simple differentiation of the integrand in (4.7) will confirm this.

The depth-integrated model above therefore provides a relatively simple means of computing the velocity field  $u(x, z)$  and hence the fluid flux  $q$  for a prescribed fluid film geometry, given by  $h$  and  $b$ . The question that remains is: what magnitude of error can we expect to incur by dropping the remainder term  $r$  in (4.1)? A complete analysis of this problem is beyond the scope of this paper, but we can sketch the basic steps required.

#### 4.2 Sketch of an asymptotic analysis of the depth-integrated model

Firstly, using the asymptotic structure of the thin-film flow problem derived in section 3, we can estimate the size of the remainder term  $r$  for each parameter regime considered in section 3. For the parameter regimes with  $\varepsilon \lesssim \lambda \lesssim 1$  in sections 3.2 and 3.3 it is straightforward to show that  $u_x = u_{b,x} + O(\varepsilon\lambda)$  and  $\tau_2 = \tilde{\tau}_2 + O(\varepsilon\lambda)$ , and hence that  $r = O(\varepsilon\lambda)$  (we can also extend this to the case considered in section 3.4: due to the rescaling there, we can treat the parameter regime considered in section 3.4 as having  $\lambda = \varepsilon$ ). For the slow sliding problem in section 3.5, estimating  $r$  is somewhat more involved. We have  $u_x = u_{b,x} + O(\varepsilon\lambda^n)$  in both, the inner and outer regions. The contribution to  $r$  of the second term in square brackets on the right-hand side of (4.2) is harder to estimate. In the outer region, we have  $\tau_1 = O(\lambda^{1-n})$  and  $\tau_2 = \tilde{\tau}_2 + O(\varepsilon\lambda^{-2}) + O(\varepsilon\lambda^{-n})$ , so the contribution of the second term in square brackets in (4.2) is of  $O(\varepsilon\lambda^{-2}) + O(\varepsilon\lambda^{-n})$ , both of which are much smaller than the error due to the first term. Likewise, in the boundary layer, we have  $\tau_1 = T_1 = O(1)$  and  $\tau_2 = \tilde{\tau}_2 + O(\varepsilon)$  with  $\tilde{\tau}_2 = O(\lambda^{-1})$ . It follows that the contribution to  $r$  due to the second term in (4.2) is of  $O(\varepsilon\lambda)$  in the boundary layer, and this is again much smaller than the  $O(\varepsilon\lambda^n)$  contribution due to the first term, as  $\lambda \gg 1$  and  $n > 1$  in the parameter regime in question. Consequently, we can estimate that  $r = O(\varepsilon\lambda^n)$  in the inner and outer regions in the slow sliding parameter regime  $1 \ll \lambda \lesssim \varepsilon^{-1/n}$ .

Secondly, we need to investigate whether the remainder term  $r$  does in fact enter into

the calculation of  $u$  to  $O(\varepsilon^2)$ . If it does not, then the depth-integrated model (4.4)–(4.6) retains the same  $O(\varepsilon^2)$  accuracy in  $u$  as Blatter’s model, but is computationally simpler to solve. If  $r$  does enter into the calculation of  $u$  to  $O(\varepsilon^2)$ , then the crucial question is: at what order? In other words, how much worse than Blatter’s model is the depth-integrated model (4.4)–(4.6)?

For the parameter regimes  $\varepsilon \lesssim \lambda \lesssim 1$ , describing intermediate and rapid sliding,  $\tau_1$  needs to be expanded to an error of  $O(\varepsilon\lambda)$  in order to compute  $u$  to an error of  $O(\varepsilon^2)$  as indicated at the ends of sections 3.2 and 3.3 (owing to the rescaling in section 3.4, we can treat the parameter range being considered there as having  $\lambda = \varepsilon$ ). It is then clear that with  $r = O(\varepsilon\lambda)$  in (4.1),  $r$  does not enter into the calculation of  $\tau_1$  to  $O(\varepsilon\lambda)$ , and hence it does not affect the calculation of  $u$  to  $O(\varepsilon^2)$ . For  $\lambda \lesssim 1$ , using the depth-integrated model (4.4)–(4.6), which omits  $r$ , therefore produces an error in  $u$  that is equivalent to the error incurred by Blatter’s model.

The slow sliding regime  $1 \ll \lambda \lesssim \varepsilon^{-1/n}$  is more delicate because of the boundary layer near the upper surface. (Owing to the rescaling in section 3.6, we can treat the parameter regime considered there as having  $\lambda = \varepsilon^{-1/n}$ .) We investigate the outer solution first. From (3.73),  $\tau_1$  needs to be expanded in the outer region as  $\tau_1 \sim \lambda^{1-n}(\tau_1^{(1)} + \dots + \lambda^{-2kn}\tau_1^{(k)} + \dots) + O(\varepsilon\lambda) \sim \lambda^{1-n}(\tau_1^{(1)} + \dots + \lambda^{-2kn}\tau_1^{(k)} + \dots + O(\varepsilon\lambda^n))$  in order to calculate  $u$  to an error of  $O(\varepsilon^2)$ , where all the retained terms  $\tau_1^{(k)}$  are such that  $\lambda^{-2kn} \gg \varepsilon\lambda^n$  (see also the comment after (3.73)). Substituting this into (4.1) and noting that  $r = O(\varepsilon\lambda^n)$ , it is immediately clear that  $r$  does not enter into the calculation of any of the  $\tau_1^{(k)}$ . In the boundary layer, on the other hand, (3.108) expands  $\tau_1 = T_1$  as  $T_1 = T_1^{(0)} + O(\varepsilon)$ . From (4.1) with  $r = O(\varepsilon\lambda^n)$ , it is clear that  $r$  *does* enter into the calculation of  $T_1^{(0)}$  as  $\lambda \gg 1$  in the parameter regime considered here; in fact, omitting  $r$  incurs an error of  $O(\varepsilon\lambda^n)$  in  $T_1^{(0)}$ . This error then propagates into the outer solution through the asymptotic matching procedure in section 3.5.3. Of course, this is only relevant when there is in fact a boundary layer: for a Newtonian fluid with  $n = 1$ , this complication does not arise (and in fact, the remainder term  $r$  in (4.2) also simplifies considerably).

To establish in detail how the error introduced by omitting  $r$  in the inner solution affects the outer solution requires an adaptation of some of the work in section 3.5.3 that is beyond the scope of the present paper. However, the result of this adaptation leads to the same conclusion as the following, simpler argument. The leading order effect of the boundary layer on the outer region is the introduction of the matching terms  $\tau_1^{(i,1)}$ ,  $u^{(i,1)}$  and  $u^{(i,2)}$  in (3.73), which feature at  $O(\varepsilon\lambda^{-2})$  in the outer solution. In fact, from (3.94) and (3.151), it is clear that an  $O(\varepsilon\lambda^n)$  error in  $T_1^{(0)}$  produces an  $O(\varepsilon\lambda^n)$  error in the function  $C_i = 2 \int_0^\infty T_{1,X}^{(0)} dZ$ , and hence an  $O(\varepsilon\lambda^n)$  error in the matching terms  $u^{(i,1)}$  and  $u^{(i,2)}$  in the outer solution. As these feature in the outer solution at  $O(\varepsilon\lambda^{-2})$ , the error in the calculation of the velocity field  $u$  introduced by omitting the remainder term  $r$  in the boundary layer is therefore of  $O(\varepsilon\lambda^{-2}) \times O(\varepsilon\lambda^n) = O(\varepsilon^2\lambda^{n-2})$ .

Recall that the parameter regime for which this error estimate was derived is  $1 \ll \lambda \lesssim \varepsilon^{-1/n}$ . It follows that the error estimate is worse than the  $O(\varepsilon^2)$  error in Blatter’s model only if  $n > 2$ ; otherwise dropping the remainder term  $r$  in (4.1) represents no loss of accuracy compared with Blatter’s model. Moreover, even for  $n > 2$ , the loss of accuracy is

relatively slight: the biggest error occurs in the limit of slow sliding,  $\lambda \sim \varepsilon^{-1/n}$ , for which  $O(\varepsilon^2 \lambda^{n-2}) = O(\varepsilon^{1+2/n})$ .

Thus the depth-integrated model (4.4)–(4.6) represents a small loss of accuracy compared with Blatter’s model, but a significant saving in terms of computational effort. When sliding is rapid ( $\lambda \lesssim 1$ ), the depth-integrated model performs just as well as Blatter’s model, while for slow sliding, it does not reproduce the high-viscosity boundary layer at the upper surface as well as Blatter’s model, leading to a slight loss of accuracy for power-law fluids with exponent  $n > 2$ .

However, in each parameter regime we have considered, the depth-integrated model (4.4)–(4.6) performs at least as well as the classical alternatives, the lubrication approximation (3.96) and the viscous membrane model (3.39): for fast sliding  $\varepsilon \lesssim \lambda \lesssim 1$ , the membrane model (3.39) is accurate to  $O(\varepsilon\lambda)$  as shown in section 3.3, while our depth-integrated model is accurate to  $O(\varepsilon^2)$ , and therefore performs at least as well as the membrane model, and better than the membrane model for  $\lambda \gg \varepsilon$ . For slow sliding  $1 \ll \lambda \lesssim \varepsilon^{-1/n}$  and  $n > 2$ , the lubrication approximation (3.96) is accurate to  $O(\varepsilon\lambda^{-2})$ , while our depth-integrated alternative is accurate to  $O(\varepsilon^2 \lambda^{n-2})$ , and therefore again performs at least as well as the lubrication approximation, or better when  $\lambda \ll \varepsilon^{-1/n}$ .

## 5. Conclusions

The higher-order glacier flow models we have considered in this paper are essentially hybrids between classical lubrication and viscous membrane models, and capture the leading order physics of thin film flows with a free upper surface, regardless of whether slip between the fluid and its base is slow or fast. Two different such hybrid models have been investigated in this paper for the flow of a shear-thinning power law fluid, one being the model due to Blatter (18) whose numerical solution requires the full fluid flow domain to be resolved, and an alternative described in section 4 that can be depth-integrated, originally due in slightly different form to (26). Both of these models perform at least as well as lubrication and viscous membrane models in the appropriate parameter regimes, but have the advantage of being applicable for both slow and fast sliding. From the practical perspective of numerical flow simulations, the depth-integrated model (4.4)–(4.6) is probably preferable to Blatter’s model despite a small loss of accuracy as it is computationally less onerous.

We reiterate that the analysis in the present paper has provided not only a theoretical justification of higher order models of the type widely used in glaciological simulations, but has also provided a more detailed picture of the asymptotic structure of solutions to thin film flow problems with shear-thinning rheologies, and in particular on the role of the high-viscosity surface boundary layer that forms at the upper, free surface of shear thinning power law fluids flowing as a thin film. This boundary layer leaves the flow problem unaffected at leading order, by analogy with the effect of the ‘weakly yielded’ region in lubrication flows with a Bingham rheology (29), but does enter into the outer problem at the next higher order of approximation if the power-law exponent is  $n \geq 2$ .

The discussion in this paper has been restricted to thin films in two dimensions to keep the paper to a more reasonable length. Adding a third dimension is straightforward, treating the additional velocity component analogously to the horizontal velocity  $u$  in the present paper: the asymptotic structure of the problem remains the same. The full form of Blatter’s model in three dimensions was given in Blatter’s original paper (18), and we do not repeat it here (see also (37)). From this, it is straightforward to repeat the steps leading up to

equation (4.4) in the present paper in order to obtain the analogue of the depth-integrated model (4.4)–(4.6) for a three-dimensional thin film; more details will be presented in a separate paper.

A more difficult extension of the model that needs to be addressed in future work is the use of a temperature-dependent viscosity, which is necessary in realistic ice sheet simulations. As an unresolved issue in ice sheet modelling in this context, we mention the ‘spoked’ temperature patterns that develop in numerical simulations of thin film flows with small thermal diffusion and significant viscous heat dissipation (44, 45, 46, 47), and which may not be adequately described by a classical lubrication approximation even in the case of no slip at the base of the fluid (45, 48). To resolve these, models that retain deviatoric normal stress gradients may be necessary (49), and higher order models may provide a starting point for modelling these temperature patterns as a thermoviscous instability analogous to thermal shear-banding.

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## References

1. G.K. Batchelor. *Introduction to Fluid Dynamics*. Cambridge University Press, Cambridge, (1967).
2. H.E. Huppert. The propagation of two-dimensional and axisymmetric viscous gravity currents over a rigid horizontal surface. *J. Fluid Mech.*, **121**, 43–58, (1982).
3. T. Erneux and S.H. Davis. Nonlinear rupture of free films. *Phys. Fluids*, **A5**(5), 1117–1122, (1993).
4. N.J. Balmforth, R.V. Craster, and R. Sassi. Dynamics of viscoplastic cooling domes. *J. Fluid Mech.*, **490**, 149–182, (2004).
5. O.K. Matar, R.V. Craster, and M.R.E. Warner. Surfactant transport on highly viscous thin films. *J. Fluid Mech.*, **466**, 85–111, (2002).
6. O.K. Matar and P.D.M. Spelt. Dynamics of thin free films with reaction driven density viscosity variations. *Phys. Fluids*, **17**(122102), doi: 10.1063/1.2130966, (2005).
7. B. Kamb, C.F. Raymond, W.D. Harrison, H. Engelhardt, K.A. Echelmeyer, N. Humphrey, M.M. Brugman, and T. Pfeffer. Glacier surge mechanism: 1982–1983 surge of Variegated Glacier, Alaska. *Science*, **227**(4686), 469–479, (1985).
8. A.C. Fowler. Sliding with cavity formation. *J. Glaciol.*, **33**(105), 255–267, (1987).
9. A.C. Fowler and D.A. Larson. On the flow of polythermal glaciers. I. Model and preliminary analysis. *Proc. R. Soc. Lond. A.*, **363**, 217–242, (1978).
10. L.W. Morland and I.R. Johnson. Steady motion of ice sheets. *J. Glaciol.*, **25**(92), 229–246, (1980).
11. K. Hutter. *Theoretical Glaciology*. D. Reidel, Dordrecht, (1983).
12. L.W. Morland. Unconfined ice-shelf flow. In C.J. van der Veen and J. Oerlemans, editors, *Dynamics of the West Antarctic Ice Sheet. Proceedings of a workshop held in Utrecht, May 6–8, 1985*, pages 99–116, Dordrecht, (1987). D. Reidel.

13. D.R. MacAyeal. Large-scale flow over a viscous basal sediment: theory and application to Ice Stream E, Antarctica. *J. Geophys. Res.*, **94**(B4), 4017–4087, (1989).
14. M. Weis, R. Greve, and K. Hutter. Theory of shallow ice shelves. *Cont. Mech. Thermodyn.*, **11**, 15–50, (1999).
15. C. Schoof. A variational approach to ice-stream flow. *J. Fluid Mech.*, **556**, 227–251, (2006).
16. C. Schoof. Marine ice sheet dynamics. Part I. The case of rapid sliding. *J. Fluid Mech.*, **573**, 27–55, (2007).
17. C. Schoof. Ice sheet grounding line dynamics: steady states, stability and hysteresis. *J. Geophys. Res.*, **112**(F03S28), doi:10.1029/2006JF000664, (2007).
18. H. Blatter. Velocity and stress fields in grounded glaciers — a simple algorithm. *J. Glaciol.*, **41**(138), 333–344, (1995).
19. F. Pattyn. Transient glacier response with a higher-order numerical ice-flow model. *J. Glaciol.*, **48**(162), 467–477, (2002).
20. B. Kamb and K.A. Echelmeyer. Stress-gradient coupling in glacier flow: I. longitudinal averaging of the influence of ice thickness and surface slope. *J. Glaciol.*, **32**(111), 267–284, (1986).
21. K. Herterich. On the flow within the transition zone between ice sheet and ice shelf. In C.J. van der Veen and J. Oerlemans, editors, *Dynamics of the West Antarctic Ice Sheet. Proceedings of a workshop held in Utrecht, May 6–8, 1985*, pages 185–202, Dordrecht, (1987). D. Reidel.
22. C.J. van der Veen. Longitudinal stress and basal sliding: a comparative study. In C.J. van der Veen and J. Oerlemans, editors, *Dynamics of the West Antarctic Ice Sheet. Proceedings of a workshop held in Utrecht, May 6–8, 1985*, pages 223–248, Dordrecht, (1987). D. Reidel.
23. A. Hubbard. The verification and significance of three approaches to longitudinal stresses in high-resolution models of glacier flow. *Geogr. Ann.*, **82A**(4), 471–487, (2000).
24. R.C.A. Hindmarsh. A numerical comparison of approximations to the Stokes equations used in ice sheet and glacier modelling. *J. Geophys. Res.*, **109**(F1), F01012, (2004).
25. F. Pattyn, L. Perichon, A. Aschwanden, B. Breuer, B. De Smedt, O. Gagliardini, G.H. Gudmundsson, R.C.A. Hindmarsh, A. Hubbard, J.V. Johnson, T. Kleiner, Y. Konovalov, C. Martin, A.J. Payne, D. Pollard, S.F. Price, M. Ruckamp, F. Saito, O. Souček, S. Sugiyama, and T. Zwinger. Benchmark experiments for higher order and full-Stokes ice sheet models (ISMIP-HOM). *The Cryosphere*, **2**, 95–108, (2008).
26. R.C.A. Hindmarsh. Qualitative dynamics of marine ice sheets. In W.R. Peltier, editor, *Ice in the Climate System*, pages 67–99. Springer-Verlag, Berlin, (1993).
27. K. Hutter. The effect of longitudinal strain on the shear stress of an ice sheet: in defence of using stretched coordinates. *J. Glaciol.*, **27**(95), 39–56, (1981).
28. R.E. Johnson and R.M. McMeeking. Near-surface flow in glaciers obeying Glen’s law. *Quart. J. Mech. Appl. Math.*, **37**, 273–291, (1984).
29. N.J. Balmforth and R.V. Craster. A consistent thin-layer theory for Bingham plastics. *J. Non-Newtonian Fluid Mech.*, **84**(1), 65–81, (1999).
30. D.R. Baral, K. Hutter, and R. Greve. Asymptotic theories of large-scale motion, temperature distribution in land-based polythermal ice sheets: A critical review and new developments. *Appl. Mech. Revs.*, **54**(3), 215–256, (2001).
31. W.S.B. Paterson. *The Physics of Glaciers*. Pergamon, Oxford, 3rd edition, (1994).

32. A.V. Wilchinsky and V.A. Chugunov. Ice stream-ice shelf transition: theoretical analysis of two-dimensional flow. *Ann. Glaciol.*, **30**, 153–162, (2000).
33. J. Colinge and J. Rappaz. A strongly nonlinear problem arising in glaciology. *M2AN*, **33**(2), 395–406, (1999).
34. R. Glowinski and J. Rappaz. Approximation of a nonlinear elliptic problem arising in a non-Newtonian fluid flow model in glaciology. *M2AN*, **37**(1), 175–186, (2003).
35. S.S. Chow, G.F. Carey, and M.L. Anderson. Finite element approximations of a glaciology problem. *M2AN*, **38**(5), 741–756, (2004).
36. J. Rappaz and A. Reist. Mathematical and numerical analysis of a three dimensional fluid flow model in glaciology. *M3AS*, **15**(1), 37–52, (2005).
37. C. Schoof. Coulomb friction and other sliding laws in a higher-order glacier flow model. *M3AS*, (in press).
38. A. Oron, S.H. Davis, and S.G. Bankoff. Long-scale evolution of thin liquid films. *Rev. Modern Physics*, **69**(3), 931–977, (1997).
39. S.M.J. Nowicki and D.J. Wingham. Conditions for a steady ice sheet - ice shelf junction. *Earth and Planetary Science Letters*, **265**, 246–255, doi:10.1016/j.epsl.2007.10.018, (2008).
40. K. Hutter, S. Yakowitz, and F. Szidarowsky. A numerical study of plane ice-sheet flow. *J. Glaciol.*, **32**(111), 139–160, (1986).
41. A.V. Wilchinsky and V.A. Chugunov. Modelling ice-divide dynamics by perturbation methods. *J. Glaciol.*, **43**(144), 352–358, (1997).
42. J.K. Dukowicz, S.F. Price, and W.H. Lipscomb. Consistent approximations for ice sheet dynamics from a principle of least action. *J. Glaciol.*, (in press).
43. D. Goldberg, D.M. Holland, and C.G. Schoof. Grounding line movement and ice shelf buttressing in marine ice sheets. *J. Geophys. Res.*, (in press).
44. A.J. Payne and P.W. Dongelmans. Self-organization in the thermomechanical flow of ice sheets. *J. Geophys. Res.*, **102**(B6), 12219–12233, (1997).
45. R.C.A. Hindmarsh. Stress-gradient damping of thermoviscous ice-flow instabilities. *J. Geophys. Res.*, **111**(B12409), doi:10.1029/2005JB004019, (2006).
46. F. Saito, A. Abe-Ouchi, and H. Blatter. European ice sheet modelling initiative (EISMINT) model intercomparison experiments with first-order mechanics. *J. Geophys. Res.*, **111**(F02012), doi:10.1029/2004JF000273, (2006).
47. E. Bueller, J. Brown, and C. Lingle. Exact solutions to the thermomechanically coupled shallow-ice approximation: effective tools for verification. *J. Glaciol.*, **53**(182), 499–516, (2007).
48. R.C.A. Hindmarsh. Thermoviscous stability of ice-sheet flows. *J. Fluid Mech.*, **502**, 17–40, (2004).
49. R.C.A. Hindmarsh. Consistent generation of ice-streams via thermo-viscous instabilities modulated by membrane stresses. *Geophys. Res. Lett.*, **36**(L06502), doi:10.1029/2008GL036877, (2009).

## APPENDIX A

*Blatter's derivation*

Blatter (18) chooses different scales for the Stokes equations (2.1)–(2.10) from those defined in section 2.1 in the present paper. Specifically, he puts

$$[\tau_1] = [\tau_2] = [p] = \rho g [h]^2 / [x], \quad [u] = A [\tau_2]^n [h], \quad [v] = [u][h]/[x].$$

Non-dimensionalizing the model with these scales (see (2.21)) produces the following as a dimensionless version of (2.2), where  $\varepsilon = [h]/[x]$  is still the aspect ratio of the fluid layer:

$$\varepsilon \tau_{1,z} + \varepsilon^2 \tau_{2,z} - \varepsilon p_z - 1 = 0, \quad (\text{A.1})$$

which is the equivalent of equation (2.24) in Blatter's paper. Immediately, it is clear that this scaling cannot be self-consistent, as (A.1) shows that either  $p = O(\varepsilon^{-1})$ ,  $\tau_1 = O(\varepsilon^{-1})$  or  $\tau_2 = O(\varepsilon^{-2})$  (i.e., not all the scales chosen for  $p$ ,  $\tau_1$  and  $\tau_2$  can be correct). The truncation at  $O(\varepsilon^2)$  rather than  $O(\varepsilon)$  in Blatter's paper glosses over this complication, which calls into question whether the error incurred in dropping terms including a factor of  $\varepsilon^2$  is indeed of  $O(\varepsilon^2)$ . To put this in a different way, Blatter's scaling does not allow a regular asymptotic expansion of the form  $\tau_1 \sim \tau_1^{(0)} + \varepsilon \tau_1^{(1)} + \dots$  etc. to be developed, as (A.1) would then suggest at leading order that  $0 = 1$ . Instead, a rescaling is required, and this is what we investigate in section 3 of the present paper.

## APPENDIX B

### *Expansion of viscosity terms*

#### B.1 *The case $\varepsilon \ll \lambda \ll 1$*

In this section, we consider the expansion of the constitutive relations (3.8) and (3.9) in the parameter regime  $\varepsilon \ll \lambda \ll 1$ , which is required in section 3.3. Given the expansions in (3.32), equation (3.8) becomes

$$\begin{aligned} u_x^{(0)} + O(\varepsilon\lambda) &= \left[ \left( \tau_1^{(0)} + \lambda^2 \tau_1^{(1)} + \dots + \lambda^{2r} \tau_1^{(r)} + \dots + O(\varepsilon\lambda) \right)^2 + \lambda^2 \left( \tau_2^{(0)} + O(\varepsilon\lambda) \right)^2 \right]^{(n-1)/2} \\ &\quad \times \left( \tau_1^{(0)} + \lambda^2 \tau_1^{(1)} + \dots + \lambda^{2r} \tau_1^{(r)} + \dots + O(\varepsilon\lambda) \right). \end{aligned} \quad (\text{B.1})$$

Expanding the squares inside the surd gives

$$\begin{aligned} &\left( \tau_1^{(0)} + \lambda^2 \tau_1^{(1)} + \dots + \lambda^{2r} \tau_1^{(r)} + \dots + O(\varepsilon\lambda) \right)^2 + \lambda^2 \left( \tau_2^{(0)} + O(\varepsilon\lambda) \right)^2 \\ &\sim \tau_1^{(0)2} + \lambda^2 \tau_2^{(0)2} + \sum_{i=1}^N \lambda^{2i} \sum_{j=0}^i \tau_1^{(j)} \tau_1^{(i-j)} + O(\varepsilon\lambda), \end{aligned}$$

if  $N$  is chosen such that  $\lambda^{2N+1} \lesssim \varepsilon \ll \lambda^{2N-1}$ . We can use this to expand the surd in a Taylor series about  $\tau_1^{(0)2}$ ,

$$\begin{aligned} &\left[ \left( \tau_1^{(0)} + \lambda^2 \tau_1^{(1)} + \dots + \lambda^{2r} \tau_1^{(r)} + \dots + O(\varepsilon\lambda) \right)^2 + \lambda^2 \left( \tau_2^{(0)} + O(\varepsilon\lambda) \right)^2 \right]^{(n-1)/2} \sim \\ &\left| \tau_1^{(0)} \right|^{n-1} + \sum_{k=1}^N \frac{1}{k!} \left( \prod_{l=1}^k \frac{n+1-2l}{2} \right) \left| \tau_1^{(0)} \right|^{n-2k-1} \left( \lambda^2 \tau_2^{(0)2} + \sum_{i=1}^N \lambda^{2i} \sum_{j=0}^i \tau_1^{(j)} \tau_1^{(i-j)} \right)^k + O(\varepsilon\lambda). \end{aligned}$$

To write this sum over powers of  $\left(\lambda^2 \tau_2^{(0)2} + \sum_{i=1}^N \lambda^{2i} \sum_{j=0}^i \tau_1^{(j)} \tau_1^{(i-j)}\right)$  as a sum of powers of  $\lambda$ , we introduce the following, recursively defined functions of the  $\tau_1^{(i)}$  and of  $\tau_2^{(0)}$ :

$$\begin{aligned} s_i &:= \begin{cases} 2\tau_1^{(0)}\tau_1^{(1)} + \tau_2^{(0)2} & i = 1, \\ \sum_{j=0}^i \tau_1^{(j)}\tau_1^{(i-j)} & i > 1, \end{cases} \\ g_{1,j} &:= s_j, \\ g_{i,j} &:= \sum_{k=1}^{j-1} g_{i-1,k} s_{j-k}. \end{aligned}$$

Then

$$\left(\lambda^2 \tau_2^{(0)2} + \sum_{i=1}^N \lambda^{2i} \sum_{j=0}^i \tau_1^{(j)} \tau_1^{(i-j)}\right)^k = \sum_{j=k}^{Nk} g_{k,j} \lambda^{2j},$$

and therefore

$$\begin{aligned} &\left[ \left(\tau_1^{(0)} + \lambda^2 \tau_1^{(1)} + \dots + \lambda^{2r} \tau_1^{(r)} + \dots + O(\varepsilon\lambda)\right)^2 + \lambda^2 \left(\tau_2^{(0)} + O(\varepsilon\lambda)\right)^2 \right]^{(n-1)/2} \sim \\ &\left| \tau_1^{(0)} \right|^{n-1} + \sum_{j=1}^N \lambda^{2j} \sum_{k=1}^j g_{k,j} \left( \frac{1}{k!} \prod_{l=1}^k \frac{n+1-2l}{2} \right) \left| \tau_1^{(0)} \right|^{n-2k-1} + O(\varepsilon\lambda). \end{aligned}$$

Armed with this, we can finally express the right-hand side of (B.1) as a sum of powers of  $\lambda$ :

$$\begin{aligned} u_x^{(0)} &= \left| \tau_1^{(0)} \right|^{n-1} \tau_1^{(0)} \\ &+ \sum_{i=1}^N \lambda^{2i} \left( \left| \tau_1^{(0)} \right|^{n-1} \tau_1^{(i)} + \sum_{j=1}^i \tau_1^{(i-j)} \sum_{k=1}^j g_{k,j} \left( \prod_{l=1}^k \frac{n+1-2l}{2} \right) \left| \tau_1^{(0)} \right|^{n-2k-1} \right) \\ &+ O(\varepsilon\lambda). \end{aligned} \tag{B.2}$$

Equating powers of  $\lambda$  on both sides of the equation, we obtain (3.34) at order  $\lambda^0$ , (3.48) at order  $\lambda^2$ , while for general  $\lambda^{2r}$ ,  $r \geq 1$ , we have

$$\begin{aligned} &\alpha_r(\tau_1^{(0)}, \dots, \tau_1^{(r)}, \tau_2^{(0)}) := \\ &\left| \tau_1^{(0)} \right|^{n-1} \tau_1^{(r)} + \sum_{j=1}^r \tau_1^{(r-j)} \sum_{k=1}^j g_{k,j} \left( \frac{1}{k!} \prod_{l=1}^k \frac{n+1-2l}{2} \right) \left| \tau_1^{(0)} \right|^{n-2k-1} = 0, \end{aligned} \tag{B.3}$$

which defines the function  $\alpha_r$  in equation (3.43).

We can also show that  $\alpha_r$  is affine in  $\tau_1^{(r)}$ , and identify the coefficient of  $\tau_1^{(r)}$ . Specifically, the definition of the  $s_i$  and  $g_{i,j}$  above implies that the only term in the sum over  $g_{k,r}$  in (B.3) that contains  $\tau_1^{(r)}$  is in fact

$$g_{1,r} = \sum_{j=0}^r \tau_1^{(j)} \tau_1^{(r-j)} = 2\tau_1^{(0)} \tau_1^{(r)} + \text{terms not containing } \tau_1^{(r)}.$$

Hence we always have

$$\alpha_r = n \left| \tau_1^{(0)} \right|^{n-1} \tau_1^{(r)} + \text{terms not containing } \tau_1^{(r)},$$

as for instance in equation (3.48). The equation  $\alpha_r = 0$  can therefore be inverted uniquely for

$\tau_1^{(r)}$  provided the coefficient of  $\tau_1^{(r)}$  does not vanish, and provided the surd in (B.1) can indeed be Taylor expanded to order  $r$ . For  $n \neq 1$ , this is the case if and only if  $\tau_1^{(0)}$  does not vanish, which we implicitly assume to be the case in section 3.3.

Applying the same approach to equation (3.9) with the expansions in (3.32), we obtain

$$u_z^{(0)} + \sum_{i=0}^N \varepsilon \lambda^{1+2i} u_z^{(i)} + O(\varepsilon^2) = \varepsilon \lambda \left| \tau_1^{(0)} \right|^{n-1} \tau_2^{(0)} + \sum_{i=1}^N \varepsilon \lambda^{1+2i} \left( \sum_{k=1}^i g_{k,i} \left( \frac{1}{k!} \prod_{l=1}^k \frac{n+1-2l}{2} \right) \left| \tau_1^{(0)} \right|^{n-2k-1} \right) + O(\varepsilon^2). \quad (\text{B.4})$$

Equating powers of  $\lambda$  on both sides, we find (3.35) at order  $\lambda^0$ , (3.49) at order  $\lambda^1$  and (3.44) at general order  $\lambda^r$  with  $r > 1$ , where

$$\beta_r \left( \tau_1^{(0)}, \dots, \tau_1^{(r)}, \tau_2^{(0)} \right) = \tau_2^{(0)} \sum_{k=1}^r g_{k,r} \left( \frac{1}{k!} \prod_{l=1}^k \frac{n+1-2l}{2} \right) \left| \tau_1^{(0)} \right|^{n-2k-1}. \quad (\text{B.5})$$

## B.2 The case $1 \ll \lambda \ll \varepsilon^{-1/n}$

We can repeat the same procedure as in section B.1 for the case of slow sliding. With the expansions (3.73), equation (3.8) becomes

$$\begin{aligned} u_x^{(0)} &+ O(\varepsilon \lambda^{-n}) + O(\varepsilon \lambda^{-2}) \\ &\sim \left[ \lambda^{-2n} \left( \tau_1^{(0)} + \lambda^{-2n} \tau_1^{(1)} + \dots + \lambda^{-2rn} \tau_1^{(r)} + \dots + O(\varepsilon \lambda^n) \right)^2 \right. \\ &\quad \left. + \left( \tau_2^{(0)} + O(\varepsilon \lambda^{-2}) + O(\varepsilon \lambda^{-n}) \right)^2 \right]^{(n-1)/2} \\ &\quad \times \left( \tau_1^{(0)} + \lambda^{-2n} \tau_1^{(1)} + \dots + O(\varepsilon \lambda^n) \right). \end{aligned} \quad (\text{B.6})$$

To apply the same method of writing the right-hand side as a sum of powers of  $\lambda$  as above, we introduce the functions

$$\begin{aligned} t_i &:= \sum_{j=0}^{i-1} \tau_1^{(j)} \tau_1^{(i-j-1)}, \\ f_{1,j} &:= t_j, \\ f_{i,j} &:= \sum_{k=0}^{j-1} f_{i-1,k} t_{j-k}. \end{aligned}$$

Then (B.6) can be written as

$$\begin{aligned} u_x^{(0)} &\sim \left| \tau_2^{(0)} \right|^{n-1} \tau_1^{(0)} \\ &\quad + \sum_{i=1}^N \lambda^{2in} \left( \left| \tau_2^{(0)} \right|^{n-1} \tau_1^{(i)} + \sum_{j=1}^i \tau_1^{(i-j)} \sum_{k=1}^j f_{k,j} \left( \frac{1}{k!} \prod_{l=1}^k \frac{n+1-2l}{2} \right) \left| \tau_2^{(0)} \right|^{n-2k-1} \right) \\ &\quad + O(\varepsilon \lambda^{-n}) + O(\varepsilon \lambda^{-2}), \end{aligned} \quad (\text{B.7})$$

if we take  $N$  such that  $\lambda^{-(2N+1)n} \gg \varepsilon \gtrsim \lambda^{-2(N+3)n}$ . Equating powers of  $\lambda$  on both sides, we obtain equation (3.77) at order  $\lambda^0$ , and equation (3.82) at general order  $\lambda^{2rn}$  with  $r \geq 1$ , where

$$\gamma_r \left( \tau_1^{(0)}, \dots, \tau_1^{(r)}, \tau_2^{(0)} \right) =$$

$$\left| \tau_2^{(0)} \right|^{n-1} \tau_1^{(r)} + \sum_{j=1}^r \tau_1^{(r-j)} \sum_{k=1}^j f_{k,j} \left( \frac{1}{k!} \prod_{l=1}^k \frac{n+1-2l}{2} \right) \left| \tau_s^{(0)} \right|^{n-2k-1}. \quad (\text{B.8})$$

It is straightforward to show that only the first term on the right-hand side of (B.8) contains  $\tau_1^{(r)}$ , so that  $\gamma_r$  is affine in  $\tau_1^{(r)}$ :

$$\gamma_r = \left| \tau_2^{(0)} \right|^{n-1} \tau_1^{(r)} + \text{terms not containing } \tau_1^{(r)}.$$

As  $\tau_2^{(0)} = -h_x(h-z)$  from (3.96), we see that the coefficient of  $\tau_1^{(r)}$  does not vanish in the outer domain (for which  $h-z \gg \lambda^{-1}$ ), and that (3.82) can be inverted uniquely for  $\tau_1^{(r)}$ . Crucially, the coefficient  $\left| \tau_2^{(0)} \right|^{n-1}$  does approach zero as  $h-z \rightarrow 0$ , underlining that  $\tau_1^{(r)}$  in the outer expansion becomes singular as  $z$  approaches  $h$ , and hence the need for a boundary layer treatment near  $z = h$ .

Applying the same approach to equation (3.9), we also recover equation (3.78) at leading order, and equation (3.87) at general order  $r \geq 1$ , where

$$\chi_r \left( \tau_1^{(0)}, \dots, \tau_1^{(r)}, \tau_2^{(0)} \right) = \tau_2^{(0)} \left( \left| \tau_1^{(0)} \right|^{n-1} + \sum_{k=1}^r f_{k,r} \left( \frac{1}{k!} \prod_{l=1}^k \frac{n+1-2l}{2} \right) \left| \tau_2^{(0)} \right|^{n-2k-1} \right).$$